ALGEBRA and FUNCTIONS

Indices:

Index (plural : indices) means power

7^3 means 7 raised to the power 3 or $7 \times 7 \times 7$

Rules of indices:

$\boxed{a^m \times a^n = a^{m+n} \qquad a^m \div a^n = a^{m-n} \qquad (a^m)^n = a^{m\times n}}$	$3^2 \times 3^3 = 3 \times 3 \times 3 \times 3 \times 3 = 3^5$ $z^3 \times z^7 = z^{10}$ $7^6 \div 7^2 = \frac{7\times7\times7\times7\times7\times7}{7\times7} = 7^4$ $(4^5)^3 = 4^5 \times 4^5 \times 4^5 = 4^{15}$
Note $\frac{a^7}{a^7} = 1$ and $a^7 \div a^7 = a^{7-7} = a^0$ so $\boxed{a^0 = 1}$ for any a ($a \neq 0$)	$6^0 = 1$ $x^0 = 1$
$\frac{1}{a^m} = 1 \div a^m = a^0 \div a^m = a^{0-m} = a^{-m}$ so $\boxed{a^{-m} = \frac{1}{a^m}}$	$7^{-2} = \frac{1}{7^2} = \frac{1}{49}$ Note: $\frac{1}{3^{-2}} = 3^2 = 9$
$a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^1 = a$ so $a^{\frac{1}{2}} = \sqrt{a}$, $a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a$ so $a^{\frac{1}{3}} = \sqrt[3]{a}$ and in general $\left(a^{\frac{1}{n}}\right)^n = a^1 = a$ so $\boxed{a^{\frac{1}{n}} = \sqrt[n]{a}}$	$9^{\frac{1}{2}} = \sqrt{9} = 3$ $64^{\frac{1}{3}} = \sqrt[3]{64} = 4$
So $\boxed{a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}} = \sqrt[n]{a^m} \quad \text{or} \quad a^{\frac{m}{n}} = (a^{\frac{1}{n}})^m = (\sqrt[n]{a})^m}$ and $a^{-\frac{m}{n}} = \frac{1}{a^{\frac{m}{n}}} = \frac{1}{\sqrt[n]{a^m}}$ or $a^{-\frac{m}{n}} = \frac{1}{a^{\frac{m}{n}}} = \frac{1}{(\sqrt[n]{a})^m}$	$4^{\frac{3}{2}} = \left(4^{\frac{1}{2}}\right)^3 = 2^3 = 8$ $8^{-\frac{2}{3}} = \frac{1}{8^{\frac{2}{3}}} = \frac{1}{\left(8^{\frac{1}{3}}\right)^2} = \frac{1}{2^2} = \frac{1}{4}$

Some numbers you should recognise as powers: $4 = 2^2$ and so $4^{½} = 2$, $8 = 2^3$ and so $8^{⅓} = 2$

Look out for:

Square numbers: 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225 Cube numbers: 8, 27, 64, 125

Powers of 2: 4, 8, 16, 32, 64, 128, 256,.......... Powers of 10: 100, 1000, ...

1. Fill in the missing powers (in the shaded areas):

(a) $16 = 2 \quad = 4$ and therefore $16 \quad = 2$ and $16 \quad = 4$

(b) $27 = 3$ and $27 \quad = 3$

(c) $64 = 8 \quad = 4 \quad = (2^2)^3 = 2$ and $8 = 64$ and $4 = 64$ and $2 = 64$

(d) $81 = 9 \quad = 3$ and $3 = 81$ and $9 = 81$

2. Complete the following:

(a) $2^{-3} = \frac{1}{\quad} =$

(b) $4^{-1} =$

(c) $125^{⅓} =$

(d) $16^{¼} =$

(e) $16^{¾} = (16^{¼})^3 = (\quad)^3 =$

(f) $32^{\frac{3}{5}} =$

(g) $4^{\frac{5}{2}} = (4^{\frac{1}{2}})^5 = (\quad)^5 =$

(h) $25^{\frac{3}{2}} = (25^{\frac{1}{2}})^3 =$

(i) $\left(\frac{1}{3}\right)^{-2} = \frac{1}{\left(\frac{1}{3}\right)^2} =$

(j) $\left(\frac{8}{27}\right)^{\frac{2}{3}} =$

(k) $\frac{1}{3^{-2}} =$

3. Simplify $\sqrt{3} \times \frac{1}{27} \times 9^{\frac{5}{2}}$ by writing each term as a power of 3

4. Simplify:

(a) $x\sqrt{x} =$ (b) $x^2\sqrt{x^3} =$ (c) $\frac{x}{\sqrt{x}} =$

(d) $\frac{x^2}{\sqrt{x^3}} =$ (e) $\frac{\sqrt{x^5}}{x^2} =$ (f) $2x^3\sqrt{x^5} =$

(g) $\frac{x^2\sqrt{x}}{2x} =$ (h) $\frac{x\sqrt{x}}{x^{-2}} =$

Surds:

Any number which cannot be expressed as a fraction is called IRRATIONAL.

Irrational numbers in the forms of roots are called <u>surds</u>. Examples are $\sqrt{3}$, $\sqrt[4]{6}$, $\sqrt[3]{10}$

Some numbers are expressed as a rational (one which <u>can</u> be written as a fraction) plus a surd. They are often left "in surd form", because this is the exact value of the number. Calculators will only give an approximate value for the number depending on how many figures appear on the readout.

So $2 + \sqrt{3}$ is the exact value, whilst your calculator might say this number is 3.732050808

You should know: $\sqrt{9} \times \sqrt{4} = 3 \times 2 = 6$ and $\sqrt{36} = 6$ so $\sqrt{9} \times \sqrt{4} = \sqrt{36}$ $\therefore \sqrt{a} \times \sqrt{b} = \sqrt{ab}$

Also $\sqrt{9} \times \sqrt{9} = 3 \times 3 = 9$ $\therefore$ $\sqrt{3} \times \sqrt{3} = 3$ and $\sqrt{2} \times \sqrt{2} = 2$ etc.

Note: $\sqrt{3} + \sqrt{2}$ cannot be added to give a single root.

Simplifying large surds.

Look for factors that are square numbers under the root sign. e.g. $\sqrt{600} = \sqrt{6 \times 100} = \sqrt{6} \times \sqrt{100} = 10\sqrt{6}$

$\sqrt{48} = \sqrt{3 \times 16} = \sqrt{3} \times \sqrt{16} = 4\sqrt{3}$

Look out for 4, 9, 16, 25, 36, 49, 64, 81, 100 and multiples of these.

Numbers involving a rational and a surd can be combined rather like algebraic brackets.

e.g. $(3 + 2\sqrt{7}) + (11 - 4\sqrt{7}) = 14 - 2\sqrt{7}$

$$(5 + 3\sqrt{5})(6 - 7\sqrt{2}) = (5 \times 6) - (5 \times 7\sqrt{2}) + (3\sqrt{5} \times 6) - (3\sqrt{5} \times 7\sqrt{2})$$

$= 30 - 35\sqrt{2} + 18\sqrt{5} - 21\sqrt{10}$ which cannot be simplified any further.

Simplify:	
5. $3(2 + 5\sqrt{2}) - 2(5 - 3\sqrt{2}) =$	*5. Expand the brackets and collect together the rational and surd parts.*
6. $(5 - \sqrt{3}) + 3(2 - \sqrt{2}) =$	*6. Beware! $\sqrt{2}$ and $\sqrt{3}$ cannot be combined by addition (or subtraction) to make one surd. Your answer will have one rational and 2 surds in it.*
7. $(3 - 2\sqrt{5})(2 + \sqrt{5}) =$	*7. Multiply out the brackets and remember $\sqrt{5} \times \sqrt{5} = 5$*
8. $(3 + \sqrt{2})^2 =$	*8. Write down the two brackets first (you are less likely to make a mistake).*
9. $(\sqrt{3} + 2\sqrt{5})(2\sqrt{3} - 3\sqrt{5}) =$	*9. Multiply out the brackets and collect the rational and surd terms. Remember $\sqrt{3} \times \sqrt{5} = \sqrt{15}$*

10. Express the following in the form $a\sqrt{b}$, where a and b are integers. (a) $\sqrt{8}$ (b) $\sqrt{18}$ (c) $\sqrt{75}$ (d) $\sqrt{360}$ (e) $\sqrt{32}$ (f) $\sqrt{27}$ (g) $\sqrt{98}$ (h) $15\sqrt{0.44}$

Rationalising a denominator:

These fractions have irrational denominators. $\frac{1}{\sqrt{3}}$, $\frac{3}{1+\sqrt{3}}$, $\frac{2}{\sqrt{3}}$, $\frac{3}{\sqrt{5}}$, $\frac{5}{1-\sqrt{11}}$, $\frac{8}{\sqrt{7}}$

To make the denominator rational you need to remember:

- Multiplying the top and bottom of the fraction by the same value results in an equivalent fraction.
- $\sqrt{a}\times\sqrt{a}=a$ (e.g. $\sqrt{4}\times\sqrt{4}=2\times2=4$ so $\sqrt{4}\times\sqrt{4}=4$ and $\sqrt{3}\times\sqrt{3}=3$ etc.)
- When you have two brackets that give the difference of two squares the answer is rational.
  For example $(1+\sqrt{2})(1-\sqrt{2})=1^2-(\sqrt{2})^2=1-2=-1$

Example: Rationalise the denominator for the following:

(a) $\frac{1}{\sqrt{3}} = \frac{1\times\sqrt{3}}{\sqrt{3}\times\sqrt{3}} = \frac{\sqrt{3}}{3}$ Multiply top and bottom of the fraction by $\sqrt{3}$ then $\sqrt{3}\times\sqrt{3}=3$ and the fraction has a rational value for the denominator.

(b) $\frac{3}{\sqrt{3}} = \frac{3\times\sqrt{3}}{\sqrt{3}\times\sqrt{3}} = \frac{3\times\sqrt{3}}{3} = \sqrt{3}$ Multiply top and bottom of the fraction by $\sqrt{3}$ then $\sqrt{3}\times\sqrt{3}=3$ and the fraction has a rational value for the denominator. Divide top and bottom by 3 to simplify.

(c) $\frac{1}{1+\sqrt{3}} = \frac{(1-\sqrt{3})}{(1+\sqrt{3})(1-\sqrt{3})} = \frac{(1-\sqrt{3})}{1-3} = \frac{(1-\sqrt{3})}{-2} = -\frac{1}{2}+\frac{\sqrt{3}}{2}$ Multiply top and bottom of the fraction by $1-\sqrt{3}$ then the fraction has a rational value for the denominator.

(d) $\frac{3-\sqrt{2}}{1+\sqrt{2}} = \frac{(3-\sqrt{2})(1-\sqrt{2})}{(1+\sqrt{2})(1-\sqrt{2})} = \frac{3-3\sqrt{2}-\sqrt{2}+\sqrt{2}\sqrt{2}}{1-2} = \frac{3-4\sqrt{2}+2}{-1}$ Multiply top and bottom of the fraction by $1-\sqrt{2}$ then the fraction has a rational value for the denominator.

$= \frac{5-4\sqrt{2}}{-1} = -5+4\sqrt{2}$

Rationalise the denominator for the following: 11. $\frac{5}{\sqrt{2}}$ 12. $\frac{12}{\sqrt{3}}$ 13. $\frac{2+3\sqrt{5}}{\sqrt{5}}$ 14. $\frac{2-\sqrt{2}}{\sqrt{2}}$	*11. Multiply the top and bottom of the fraction by $\sqrt{2}$ Here you are "rationalising the denominator". In other words the surds are removed from the denominator, making the fraction easier to deal with. Similar idea for 12, 13 and 14.*
Example $\frac{2+\sqrt{7}}{3-\sqrt{7}} = \frac{2+\sqrt{7}}{3-\sqrt{7}}\times\frac{3+\sqrt{7}}{3+\sqrt{7}}$ $= \frac{(6)+(2\sqrt{7})+(3\sqrt{7})+7}{(3)^2-(\sqrt{7})^2}$ $= \frac{13+5\sqrt{7}}{9-7} = \frac{13+5\sqrt{7}}{2} = \frac{13}{2}+\frac{5\sqrt{7}}{2}$	*Multiply numerator and denominator by $3+\sqrt{7}$. This will give 'the difference of two squares' in the denominator and will make it rational. Since the top and bottom of the fraction are multiplied by the same amount, it is equivalent and it can be tidied up more easily.*
15. Rationalise the denominator $\frac{3+\sqrt{5}}{4+\sqrt{3}}$	*Since the denominator is $4+\sqrt{3}$, you multiply numerator and denominator by $4-\sqrt{3}$*
16. Simplify this expression by rationalising the denominator $\frac{4}{\sqrt{5}+2}$	*The denominator is $\sqrt{5}+2$ so the multiplier is $\sqrt{5}-2$*

Terms: A term is a collection of numbers, letters, functions and brackets all multiplied together. Terms can be combined using + and – signs to produce an expression.

E.g. $4x^2 - 5ax + 3x - 6$

Collecting like terms: '*like terms*' have exactly the same combination of letters e.g. x terms or x^2y terms. *Like terms* can be added or subtracted. e.g. $3x^2 + 5xy - 4x - 6x^2 + 2xy - 6x - 4 = -3x^2 + 7xy - 10x - 4$

Multiplying out brackets:

Remove the brackets and simplify the following:	
17. $5(3x-7)$	*17. Multiply each term in the bracket by 5.*
18. $-(2x-3)$	*18. Careful: $-(2x-3)$ means $-1(2x-3)$*
19. $-2(6x+11)$	
20. $(2x+7)(3x-5)$	*20. Multiply each term in the first bracket by each term in the second bracket in your head. Simplify.*
21. $(x-4)(2x-5)$	*21. Careful with the minuses.*
22. $(2x-3)^2$	*22. Always write out as two brackets and continue as before.*
23. $(x^2-3x+4)(2x^2-5x+1)$	*23. Just more terms! Be systematic: x^2 times all the terms in the second bracket, then $-3x$ times all the terms etc. Be careful with the minus terms. Remember to collect together like terms.*
24. $(x+2)(x-5)(x-1)$	*24. Multiply out two brackets first. Then multiply the result by the remaining bracket.*

Identities:

An identity is another identical way of writing an expression. It is true for all values of the variable(s). The coefficient of each term will be equal.

Example: Consider $x^2 - 6x + 5 \equiv (x+a)^2 + b$

$\equiv x^2 + 2ax + a^2 + b$ is true for all x

Whereas $x^2 + 4x = 12$ is an equation and is only true for two values of x.

equating coefficients of x: $-6 = 2a \Rightarrow a = -3$

equating constant terms: $5 = a^2 + b \Rightarrow 5 = 9 + b \Rightarrow b = -4$

$\therefore\ x^2 - 6x + 5 \equiv (x-3)^2 - 4$

25. Find A and B if $2x + 1 \equiv \text{A}(x+1) + \text{B}(x-2)$ Complete: $\equiv$ equating coefficients of x: $=$ equating constant terms: $=$	Multiply out the right hand side. Solve the two equations to find A and B.
26. Find a and b if $2x^2 + 8x + 5 \equiv 2(x+a)^2 + b$	*Expand the right hand side and equate coefficients.*

Factorising expressions:
Look at each term and see what factors (letters or numbers) are common to each term.
Write down these factors and then brackets. Inside the brackets, write down the terms that when multiplied by the factor/s outside give the equivalent expression.

27. Factorise:
(a) $12xy + 4x$ (b) $8x - 12$ (c) $24ab^2 - 15a^2b$

(d) $6abc + 8bc$ (e) $9p^2q^2 - 6pq^3$ (f) $z^2 - z$

28. Factorise:
(a) $6x + 9$ (b) $12x - 9$ (c) $x^2 - 5x$ (d) $2x^2 + 8x$

Sometimes the terms of an expression will have a common factor(s) that are brackets. Just treat the bracket like a letter/symbol and take it out as a factor.
e.g. $6(x+1)y + 4(x+1)z$ rather than multiply out and then try to factorise, you can think of it as
$6ay + 4az$ which factorises to $2a(3y + 2z)$
So $6(x+1)y + 4(x+1)z = 2(x+1)(3y+2z)$
Also note that when dealing with brackets $(x-3) = -(3-x)$ and $(3-x) = -(x-3)$ and so on with other brackets. You are taking -1 out as a factor.

29. Factorise:
(a) $x(x-1) + 5(x-1)$ (b) $3(x+2)(x-1) - (x-3)(x+2)$

(c) $2x^2(x+1) + 8x(x-1)(x+1)$ (d) $(x+4)(1-x) - x(x-1)(x+3)$

Notice how it is much easier to take out brackets than it would be to multiply out the brackets and then try to factorise the resulting quadratic or cubic.

Quadratic expressions:
These are expressions of the form $ax^2 + bx + c$ (Note: not an equation to be solved.)

Factorising quadratic expressions:
You may wish to do this in order to solve an equation or to help you draw a graph.

Type 1 No constant term. Example. Factorise $x^2 + 5x$ $x^2 + 5x = x(x+5)$	*Take out an x as a factor.* *See what to multiply it by to make it equal to the original expression.*
Type 2 No x term. Here it will be the ***difference of two squares*** and will factorise easily, or it will be a sum and you will not be able to factorise it. Example. Factorise $x^2 - 4$ $x^2 - 4 = (x+2)(x-2)$	*Here the x s multiply to give x^2, the +2 and the –2 multiply to give the – 4 and the +2 times x and the –2 times x add to give 0.*
Type 3 All terms included. Not all quadratic expressions factorise but if you need to do it in an exam - it will! (Note: Solving a quadratic equation is different - you can use the quadratic formula if it does not factorise.) Example. Factorise $x^2 + 7x + 12$ $x^2 + 7x + 12 = (x+3)(x+4)$	*Here the x s multiply to give x^2, the +3 and the +4 multiply to give the +12 and the +3 times x and the +4 times x add to give 7x*

30. Factorise $x^2 - 3x$	31. Factorise $2x^2 - 6x$	32. Factorise $x^2 - 7$
33. Factorise $x^2 - x - 20$	34. Factorise $x^2 + 7x + 12$	35. Factorise $x^2 - 10x + 16$
36. Factorise $3x^2 + 2x - 8$		*$3x$ and x give $3x^2$ and try pairs of factors of -8 to see which will give $+2x$ when multiplied out and added.*
37. Factorise $6x^2 - 5x - 1$		*$3x$ and $2x$ OR $6x$ and x give $6x^2$* *Try pairs of factors of -1 to see which will give $-5x$ when multiplied out and added.*
38. Factorise $8x^2 - 2x - 3$	39. Factorise $12x^2 + 4x - 5$	
40. Factorise $9x^2 - 16$	41. Factorise $4x^2 - 121$	

Algebraic fractions: $+ - \times \div$ and 'cancelling down' algebraic fractions:

These techniques will often help to simplify expressions or make them easier to work with.
The same rules apply as with numbers.
If you multiply or divide the numerator and denominator of a fraction by the same value, then you obtain an equivalent fraction. When dividing top and bottom by the same value, it is often called cancelling down.

Number	Algebra
$\frac{2+3}{8}$ the 2 and 8 <u>do not cancel</u>. When cancelling you must think 'divide all the top and all the bottom by ..' Obviously this $= \frac{5}{8}$ (If you cancelled you would get $\frac{1+3}{4} = \frac{4}{4} = 1$) $\frac{2+6}{12} = \frac{{}^1\cancel{2}+{}^3\cancel{6}}{{}^6\cancel{12}} = \frac{4}{6} = \frac{2}{3}$ (Dividing all the top by 2 and the bottom by 2. The answer is correct as $\frac{8}{12} = \frac{2}{3}$)	$\frac{x^2+3}{2x}$ the x and x^2 <u>do not cancel</u>. When cancelling you must think 'divide **all** the top and **all** the bottom by ..' Be careful when there are + or – in the fractions and you think about cancelling. $\frac{2x+4}{8x} = \frac{x+2}{4x}$ Dividing all the top by 2 and the bottom by 2. (The x does not 'cancel' If not sure put in a number for x.) or factorise $\frac{2x+4}{8x} = \frac{2(x+2)}{8x} = \frac{{}^1\cancel{2}(x+2)}{{}^4\cancel{8}x} = \frac{x+2}{4x}$ Sometimes you can cancel brackets $\frac{6x+3}{2x+1} = \frac{3\cancel{(2x+1)}}{\cancel{2x+1}} = 3$
$\frac{2}{3} \times \frac{7}{8} = \frac{{}^1\cancel{2}}{3} \times \frac{7}{{}^4\cancel{8}} = \frac{7}{12}$ (You can cancel one number on the top with one number on the bottom **when multiplying**, as $\frac{14}{24} = \frac{7}{12}$)	$\frac{2}{x} \times \frac{5x}{2x^2+4} = \frac{2}{x} \times \frac{5x}{2(x^2+2)} = \frac{{}^1\cancel{2}}{{}^1\cancel{x}} \times \frac{5\cancel{x}}{\cancel{2}(x^2+2)} = \frac{5}{x^2+2}$ factorise cancel
$\frac{3}{4} \div \frac{6}{7} = \frac{3}{4} \times \frac{7}{6} = \frac{{}^1\cancel{3}}{4} \times \frac{7}{{}^2\cancel{6}} = \frac{7}{8}$	$(x^2 - 5x) \div \frac{x}{2} = \frac{x^2-5x}{1} \div \frac{x}{2} = \frac{\cancel{x}(x-5)}{1} \times \frac{2}{\cancel{x}} = 2(x-5)$
$\frac{5}{6} + \frac{7}{15} = \frac{25+14}{30} = \frac{39}{30} = 1\frac{9}{30}$	$\frac{3}{x} + \frac{2}{x^2-5x} = \frac{3}{x} + \frac{2}{x(x-5)} = \frac{3(x-5)}{x(x-5)} + \frac{2}{x(x-5)}$ factorise; common denominator - times top and bottom of the first fraction by $(x - 5)$ $= \frac{3x-15+2}{x(x-5)} = \frac{3x-13}{x(x-5)}$ tidy up *This cannot be made simpler, as there are no factors common to both the numerator and the denominator.*

Simplify these:

42. $\dfrac{3x+9}{x^2-9} =$	*The top and the bottom both factorise. The bottom is the difference of two squares i.e. (x + 3)(x – 3). Then cancel.*
43. $\dfrac{2}{x-2}+\dfrac{1}{x+1} =$	*Obtain the same denominator by multiplying top and bottom of 1st fraction by (x + 1) and multiplying top and bottom of the 2nd fraction by (x – 2).*
44. $\dfrac{2x^2-10x}{6x-30} =$	*Factorise the top and the bottom. Then cancel.*
45. $\dfrac{\frac{1}{2}(x-2)}{x^2-4} =$	*Get rid of the fraction within the fraction by multiplying the top and bottom by 2. Careful - you may want to factorise the bottom first.*
46. $\dfrac{x^2+x-6}{4x^2-6x-4} =$	*Best to take 2 out as a factor in the denominator first, then factorise the top and bottom. Then cancel.*
47. $(x+4)\times\dfrac{3}{x^2-16} =$	*The x + 4 is multiplied by the top only (as x + 4 = $\frac{x+4}{1}$). Factorise the $x^2 - 16$.*
48. $\dfrac{\frac{x+2}{x+3}}{3x+6} =$	*Multiply top and bottom of fraction by x + 3 OR use $\frac{3x+6}{1}$ for the denominator and then rule for ÷*
49. $\dfrac{2x-6}{\frac{2x}{x+3}} =$	*Multiply top and bottom of fraction by x + 3 OR the numerator $\frac{2x-6}{1}$ ÷ by the denominator. Then rule for ÷*
50. $\dfrac{x-2}{3}+\dfrac{x+1}{4} =$	*× top and bottom of the 1st fraction by 4 and × top and bottom of the second fraction by 3 to obtain a common denominator.*
51. $\dfrac{2}{x+1}+\dfrac{1}{(x+1)^2} =$	*× the top and bottom of the 1st fraction by x+1 to obtain a common denominator.*
52. $\dfrac{2}{3(x+4)}-\dfrac{3}{4(x-1)} =$	*× top and bottom of the 1st fraction by 4 (x – 1) and × top and bottom of the second fraction by 3 (x + 4) to obtain a common denominator.*
53. $2+\dfrac{3}{x^2} =$	*2 is the same as $\frac{2}{1}$. × top and bottom of this fraction by x^2 to obtain a common denominator.*
54. $x-\dfrac{2}{2x+1} =$	*Write the x as a fraction. Then × the top and bottom of the 1st fraction to obtain a common denominator.*
55. $\dfrac{1}{2x}+\dfrac{1}{x^2} =$	*× top and bottom of the 1st fraction and × top and bottom of the second fraction to obtain a common denominator.*
56. $1+\dfrac{1}{x}+\dfrac{1}{x^2} =$	*This time just 3 fractions. Make the $1=\frac{1}{1}$ then obtain a common denominator for all three fractions.*
57. $\dfrac{1}{1+x^2}+\dfrac{1+x^2}{1-x^2} =$	*Just being a bit awkward. Still use the same ideas.*

Completing the square:

Example: Complete the square for $x^2 + 8x - 5$

$$x^2 + 8x - 5 = (x + 4)^2 - 16 - 5$$
$$= (x + 4)^2 - 21$$

x plus half the coefficient of x in the bracket - squared - minus half the coefficient of x squared (– 16) - then put in the constant term. Tidy up.

58. Complete the square for $x^2 - 4x + 3$

Careful - half the coefficient of x is –2.

Example: Complete the square for $2x^2 + 6x + 1$

$$2x^2 + 6x + 9 = 2(x^2 + 3x + ½)$$
$$= 2[(x + \tfrac{3}{2})^2 - \tfrac{9}{4} + \tfrac{1}{2}]$$
$$= 2[(x + \tfrac{3}{2})^2 - \tfrac{7}{4}]$$
$$= 2(x + \tfrac{3}{2})^2 - \tfrac{7}{2}$$

First take out 2 as a factor. Complete the square on the rest.

Tidy up.

Multiply by the 2.

59. Complete the square for $3x^2 - 12x + 5$

60. Complete the square for $2x^2 + 10x - 5$

61. Factorise and complete the square for the following quadratic expressions:

(a) $x^2 + 6x - 7$

(b) $4x^2 + 4x - 3$

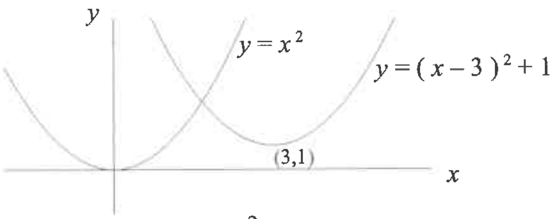
Consider: $x^2 - 4x + 5 = (x - 2)^2 - 4 + 5 = (x - 2)^2 + 1$

The bracket squared is **always** positive or zero, whatever the value of x, and therefore has a minimum value of zero. In this example you are adding 1. So the whole expression $x^2 - 4x + 5$ is always positive whatever the value of x and has a minimum value of 1. Graphically the curve $y = x^2 - 4x + 5$ is above the x axis.

Completing the square and transformations:

$y = (x + p)^2 + q$ is a translation of $y = x^2$ by the vector $\begin{bmatrix} -p \\ q \end{bmatrix}$

e.g.



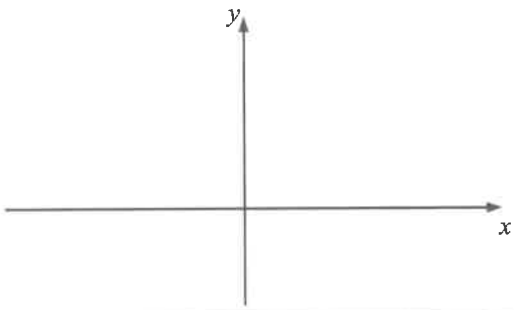
Line of symmetry of $y = (x - 3)^2 + 1$ is $x = 3$

If you have completed the square to obtain $k(x + p)^2 + q$

The vertex will be at $\begin{bmatrix} -p \\ q \end{bmatrix}$

(The $y = x^2$ graph will be stretched in the y direction by a factor k and then translated by the above vector.)

62. Complete the square and hence sketch the curve for
$y = x^2 - 7x + 10$



As before.

Label the vertex (the minimum point). (This can also be found by differentiation.)
Note: The constant in the original quadratic gives the intercept on the y axis. (Where x = 0)

63. Complete the square and hence find the coordinates of the turning point for the curve $y = -x^2 - 6x + 10$

First take the – 1 out as a factor. Complete the square on the remaining factor and then multiply by the – 1.

Solving quadratic equations:
These are equations where the highest power of x is 2. E.g. $x^2 + 8x - 5 = 0$, $x^2 + x = 2$, $x^2 = 3x$

It is best to rearrange into the form $\boxed{ax^2 + bx + c = 0}$ (Type 2 can be an exception.)

Type 1– no constant term. Example: Solve $x^2 + 8x = 0$ $x(x+8) = 0$ either $x = 0$ or $x + 8 = 0$ $\therefore x = 0$ or -8	*Factorise left hand side. (Take x out as a factor.)* *Two factors multiply to give 0 therefore one must be 0. (You can miss out this line if you want.) Note: could be solved by using the quadratic formula.*
Type 2 – no x term. Example Solve $x^2 - 6 = 0$ $x^2 = 6$ $x = \sqrt{6}$ or $-\sqrt{6}$	*For this type only.* *Rearrange to give $x^2 =$ ....* *Then square root to give two answers. (Again the quadratic formula could be used.)*
Type 3 – all terms. Example: Solve $x^2 + 3x = 4$ $x^2 + 3x - 4 = 0$ $(x-1)(x+4) = 0$ either $x - 1 = 0$ or $x + 4 = 0$ $\therefore x = 1$ or -4	*Rearrange to give = 0.* *Factorise if possible. (You will have to use the 'quadratic formula' or 'completing the square' if it does not factorise, or you are unable to factorise it.)* *Two factors multiply to give 0 therefore one must be 0.* *Hence find the two values of x.*

64. Solve $2x^2 = 5x$ *Important - do not divide by x as this removes one of the solutions.*	65. Solve $4x^2 - 7 = 0$	66. Solve $x^2 = 7x - 10$
67. Solve $2x^2 - x - 6 = 0$	68. Solve $x^2 = x$	69. Solve $x^2 + 3 = 5$

If the quadratic equation does not factorise (or you cannot find the factors) but it has real solutions, you can solve it by completing the square, or by using the quadratic formula.

Using completing the square:

70. Solve the equation $x^2 - 10x + 5 = 0$ by completing the square. Give your answer in surd form.	*Complete the square on the left hand side.* *Bracket squared on the left hand side number on the right.* *Square root both sides. The right hand side will then be $\pm$ and the root can be written in the form $a\sqrt{b}$* *Add 5 to both sides so that you obtain $x =$.........*

71. Solve the equation $2x^2 + 12x - 7 = 0$ by completing the square. Give your answer in surd form.	*First divide throughout by 2. Then proceed as above. Complete the square on the left hand side.* *Bracket squared on the left hand side number on the right.* *Square root both sides. Rationalise the denominator.* *Obtain $x =$.........*

Using the quadratic formula:
If rearranged into the form $ax^2 + bx + c = 0$ a quadratic equation can be solved using:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Note: This formula is derived by first completing the square on $ax^2 + bx + c = 0$ and then rearranging the result to give x. You could try to obtain the formula yourself as algebraic practice.

72. Solve $x^2 - 2x = 5$ giving your answer in simplified surd form. Complete: $x^2 - 2x - 5 = 0$ $a = 1$, $b = -2$, $c = -5$ $x =$ $=$ $=$	*In surd form - the 'clue' to using the formula (or by completing the square). Rearrange to give = 0.* *Careful with minus signs.* *Substitute into formula.* *The exact answers. Simplify.* *Note:* $\sqrt{24} = \sqrt{4 \times 6} = \sqrt{4}\,\sqrt{6} = 2\sqrt{6}$
73. Solve $2x^2 = 4 - 3x$ giving your answer in surd form.	*Rearrange to give = 0.* *Write out a, b, c (you don't need this but you are less likely to make a mistake).* *Substitute into the formula.* *Tidy up to give the exact answers. (No calculator!)*

74. Solve $2x^2 - 3x - 6 = 0$ giving your answer to 2 decimal places.	75. Solve $12x - 7 = 3x^2$ giving your answer to 2 decimal places.

Solving other quadratic equations:

76. Find both solutions of the equation $x + 2 = \dfrac{15}{x}$ Complete: $= 15$ $= 0$ $=$ $x =$ or	*The question says there are two solutions so you must obtain a quadratic.* *Multiply both sides by x.* *Rearrange the quadratic to give = 0.* *Solve. (This does factorise but you could use the quadratic formula.)*

77. Solve $\dfrac{x+1}{2x-1}=\dfrac{2x+3}{x-1}$ giving your answer to 3 significant figures.	*Here you could rearrange to give = 0 and then subtract the fractions.* *Multiply both sides by (2x – 1) and (x – 1). This gets rid of the fractions.* *Multiply out the brackets.* *Rearrange the quadratic to give = 0.* *Solve using the quadratic formula. Look it up and learn it if you do not know it!*
78. Solve $\dfrac{2}{x+1}+\dfrac{3}{x-2}=2$ giving your answer to 3 significant figures.	*Add the fractions.* *Multiply both sides by the common denominator to obtain a quadratic in x. Rearrange to give = 0 and solve using the quadratic formula.*
79. Solve $\dfrac{1}{(x+2)^2}=4$	*Multiply both sides by* $(x+2)^2$ *Multiply out the brackets. Careful - best to write down two brackets and multiply them out.* *Rearrange to give = 0 and solve.*
80. If $f(x)=\dfrac{x}{x+3}$ and $g(x)=\dfrac{3}{x-2}$, find the values of x for which $f(x) = g(x)$ giving your answer to 2 d.p.	*Put the algebraic fractions equal to each other and solve as before.*

Quadratic equations in disguise:

You need to be able to recognise and solve equations which are quadratic in some function of x

e.g. $x^4 - 13x^2 + 36 = 0$ which can be considered as $(x^2)^2 - 13(x^2) + 36 = 0$ i.e. a quadratic in x^2

$x^{\frac{2}{3}} - 5x^{\frac{1}{3}} + 6 = 0$ which can be considered as $\left(x^{\frac{1}{3}}\right)^2 - 5\left(x^{\frac{1}{3}}\right) + 6 = 0$ i.e. a quadratic in $x^{\frac{1}{3}}$

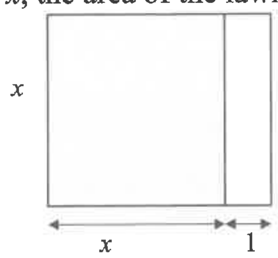
Example 1: Solve $x^4 - 13x^2 + 36 = 0$ $(x^2)^2 - 13(x^2) + 36 = 0$ Let $t = x^2$ $\therefore$ $t^2 - 13t + 36 = 0$ $(t-4)(t-9) = 0$ $t = 4$ or 9 $x^2 = 4$ or $x^2 = 9$ $x = \pm 2$ or $x = \pm 3$ i.e. $x = \pm 2$ or ± 3	*Rewrite as a quadratic in* x^2 *Let* $t = x^2$ *and write as a quadratic in t. Solve for t.* *Replace t with* x^2 *and solve the two equations.*

Example 2: Solve $x^{\frac{2}{3}} - 5x^{\frac{1}{3}} + 6 = 0$ $\left(x^{\frac{1}{3}}\right)^2 - 5\left(x^{\frac{1}{3}}\right) + 6 = 0$ Let t = $x^{\frac{1}{3}}$ $\therefore$ $t^2 - 5t + 6 = 0$ $(t-2)(t-3) = 0$ $t = 2$ or 3 $x^{\frac{1}{3}} = 2$ or $x^{\frac{1}{3}} = 3$ $x = 8$ or $x = 27$ i.e. $x = 8$ or 27	*Rewrite as a quadratic in* $x^{\frac{1}{3}}$ *Let* $t = x^{\frac{1}{3}}$ *and write as a quadratic in t.* *Solve for t.* *Replace t with* $x^{\frac{1}{3}}$ *and solve the two equations.*
81. Solve $x^4 - 2x^2 - 3 = 0$	*Rewrite as a quadratic in* x^2 *Let* $t = x^2$ *and write as a quadratic in t.* *Solve for t.* *Replace t with* x^2 *and solve the two equations.*
82. Solve $y^2 - 12y + 20 = 0$ Hence find the solutions of $(x^2+1)^2 - 12(x^2+1) + 20 = 0$	*Solve the quadratic in y.* *Replace y with* $x^2 + 1$ *and solve the two equations. You should obtain 4 answers for x.*
83. Solve $x^{\frac{2}{3}} - 7x^{\frac{1}{3}} - 8 = 0$	*Rewrite as a quadratic in* $x^{\frac{1}{3}}$ *Let* $t = x^{\frac{1}{3}}$ *and write as a quadratic in t.* *Solve for t.* *Replace t with* $x^{\frac{1}{3}}$ *and solve the two equations.*
84. Solve $2x = 7\sqrt{x} - 3$	*This is a quadratic in* $\sqrt{x} = x^{\frac{1}{2}}$ *Rearrange to = 0* *Rewrite as a quadratic in* $x^{\frac{1}{2}}$ *Let* $t = x^{\frac{1}{2}}$ *and write as a quadratic in t. Solve for t.* *Replace t with* $x^{\frac{1}{2}}$ *and solve the two equations.*

Solving problems using quadratic equations:

Just write down formulas given in words, then equate or substitute and rearrange to give a quadratic to solve.

85. A square lawn of side x metres, has a path 1 metre wide at one end as shown. Show that if the area of the path is equal to one tenth the area of the lawn then $x^2 - 10x = 0$. Hence find x, the area of the lawn and the area of the path.



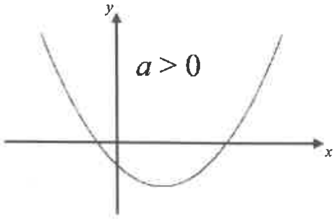
86. The sides of a right angled triangle are x, $3(x+1)$ and $3x+4$ By forming a quadratic equation and solving it, find the side lengths.

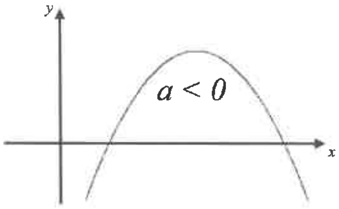
A sketch should help. Use Pythagoras.

The discriminant of a quadratic:

Using the formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ you can see that there are no solutions if $b^2 - 4ac < 0$ (i.e. You obtain a negative square root.) $\boldsymbol{b^2 - 4ac}$ is called the **DISCRIMINANT** of the quadratic.
You can 'see' where there are solutions (roots) by considering the graph of $y = ax^2 + bx + c$
(Putting $y = 0$ and either factorising or using the formula, gives you where the curve cuts/touches the x axis.)

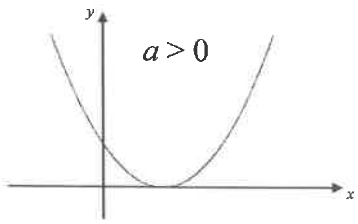
$\boldsymbol{b^2 - 4ac > 0}$

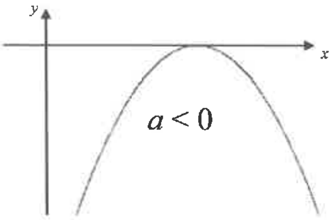




Two real distinct (different) roots.

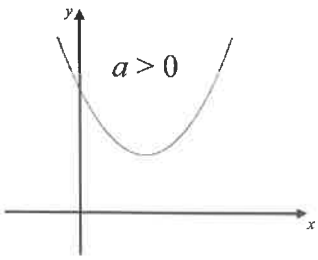
$\boldsymbol{b^2 - 4ac = 0}$

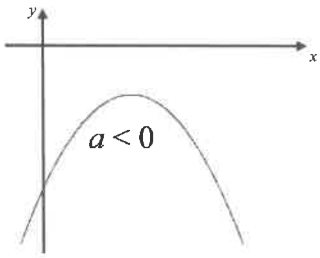




Two real equal roots (a double root).

$\boldsymbol{b^2 - 4ac < 0}$





No real roots.

87. Without working out the solutions, find out whether the following equations have either two distinct solutions, or two equal solutions, or no real solutions.
(a) $2x^2 + x - 8 = 0$ (b) $x^2 - 4x + 4 = 0$ (c) $3x^2 - 5x + 2 = 0$ (d) $x^2 + 3x + 5 = 0$

88. Find k if $2x^2 + kx + 8 = 0$ has two real equal roots.

For real equal roots $b^2 - 4ac = 0$. Use this with $a = 2$, $b = k$ and $c = 8$

Rearrange to give k^2 =..........

Hence two possible answers for k.

Sketching quadratic curves:

To sketch a quadratic curve $y = ax^2 + bx + c$

- You know $+x^2$ is $\cup$ shaped and $-x^2$ is $\cap$ shaped.
- You can use completing the square (or differentiation - covered later) to find the position of the vertex.
- To find where it cuts the y axis you put $x = 0$ into the equation.
- To find where it cuts the x axis (if it does) you put $y = 0$ into the equation and solve the resulting quadratic equation. If no real solutions ($b^2 - 4ac < 0$) then the curve is wholly above, or wholly below the x axis.

89. Sketch the curve $y = x^2 + 6x + 5$

Complete the square to find the vertex.

$y = x^2 + 6x + 5$

$y = (\qquad)^2 - \qquad +$

$y = (\qquad)^2 -$

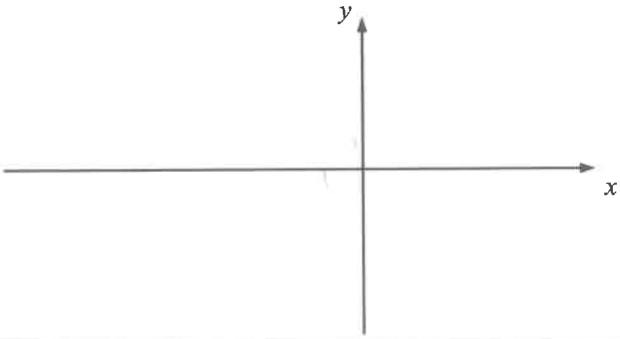
minimum point (,)

Point where curve cuts y axis $x = 0$ $y =$

Points where curve cuts x axis. $y = 0$, $0 = x^2 + 6x + 5$

This one factorises. $= (\qquad)(\qquad)$

$x =$ or



Use a ruler for the axes and label. No need to scale the axes for a sketch. –5 and the –1 just need to be in approximate positions relative to each other and the axes.

Sketch curve. Label where the curve cuts both axes and the vertex.

90. Sketch the curve $y = x^2 + 4x + 3$

91. Sketch the curve $y = x^2 - x - 2$

Rearranging formulas:

This is changing the subject of a formula. You need to: clear fractions; remove brackets; collect terms together (and take the subject out as a factor if in more than one term) ; divide.

92. Make a the subject of the formula $s = ut + \frac{1}{2}at^2$

Multiply all terms by *2.*

Subtract 2ut.

Divide by t^2

93. Make m the subject of the formula $n = \dfrac{3-m}{m+2}$

Multiply by m + 2.

Expand bracket.

Collect terms with m in on one side the rest on the other.

Take the m out as a factor.

Divide to give m.

94. Make x the subject of the formula $a\sqrt{x} + b = \sqrt{x} + c$	*Collect terms with x in on one side the rest on the other.* *Take $\sqrt{x}$ out as a factor.* *Divide by the bracket.* *Square both sides.*
95. Complete the square on $x^2 - 4x + 10$ and hence make x the subject of the equation $y = x^2 - 4x + 10$	*Complete the square.* *Write the equation with completed square form on the right hand side.* *Proceed as with solving a quadratic equation using completing the square.* *Subtract 6 from both sides.* *Square root both sides. Don't forget the ±* *Add 2 to both sides.*

Simultaneous equations:

If both equations are linear, there are three possible methods:

- drawing graphs of both equations and finding the point of intersection *(not covered here – only do this if asked!)*
- elimination
- substitution.

Elimination.

96. Solve $2x + 3y = 6$ …………(i) $x - 2y = -11$……….(ii)	*Multiply (ii) by 2 to give the same number of xs.* *Subtract. (Careful – – makes +)* *Divide and find y.* *Substitute in (i) or (ii) to find x.*
97. Solve $2x - 2y = 10$ $3x + y = 7$	98. Solve $3x - 4y = 14$ $4x - 5y = 17$

Substitution.

99. Solve $3x - y = 1$ …………(i) $4x + 3y = 10$ …………(ii)	*Rearrange (i) to give y = ………..* *Substitute in (ii)* *Expand bracket.* *Collect like terms.* *Find x.* *Substitute back into (iii) to find y.*

100. Solve $3x - 2y = 13$ $3x + y = 7$	101. Solve $x - 3y = 5$ $2x - 5y = 8$

One linear, one not - solve by substitution method.

102. Solve $2p + q = 1$ ....(i) $pq + q^2 = 3$ ....(ii)	*Using linear equation find q =* *Substitute into (ii).* *Multiply out. Careful - you may wish to multiply out the ( )^2 as two brackets ( )( ) =* *Collect together like terms.* *Solve the quadratic. (It factorises.) This gives two values for p.* *Find the corresponding values of q.*

103. Solve $2y = x + 5$ $x^2 + y^2 = 25$	104. Solve $2x + y = 2$ $y = x^2 + x - 2$

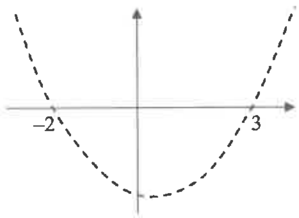
Inequalities:
Manipulate as with equations, except when multiplying or dividing by a negative number, when you need to change the inequality sign around. (Or make sure that you do not have to multiply, or divide, by a negative number by careful manipulation. Note: $4 > x$ is the same as $x < 4$ etc.)

105. Solve $2x - 3 < 5x + 2$	*xs on one side, numbers on other.* *Collect together.* *Divide by –3 to get x. Don't forget to change the inequality sign.*

Quadratic inequalities:

To solve $x^2 - x - 6 < 0$ factorise $(x-3)(x+2) < 0$

Sketch $y = (x-3)(x+2)$



Use the convention for GCSE inequalities – dotted line for > or < and solid line for ≤ or ≥ **You should be able to shade a region given inequalities.**

You can see y is negative for values of x between – 2 and 3.

So $(x-3)(x+2) < 0$ for $-2 < x < 3$ Using set notation the solution is written $\{x : -2 < x < 3\}$
or $\{x : x > -2\} \cap \{x : x < 3\}$

Read as ' such that'

The first bracket AND the second bracket have to hold true at the same time.

Similarly $(x-3)(x+2) \geq 0$ for $x \leq -2$ or $x \geq 3$ Using set notation the solution is written

$$\{x : x < -2\} \cup \{x : x > 3\}$$

The first bracket OR the second bracket have to hold true.

Interval notation:

You may not have covered this specifically for your course, but it is useful to know.

The interval $[2, 5)$ is equivalent to $2 \leq x < 5$
The interval $[2, 5]$ is equivalent to $2 \leq x \leq 5$
The interval $(2, 5]$ is equivalent to $2 < x \leq 5$
The interval $[2, \infty)$ is equivalent to $x \geq 2$ and so on.

106. Find the set of values of x for which $x^2 + 3x + 2 \geq 0$

Need to sketch the graph of $y = x^2 + 3x + 2$.

First find where it cuts the x axis. i.e. when $y = 0$.

Factorise.

Find x.

Sketch $y = x^2 + 3x + 2$.

What values of x make the y coordinates $\geq$ 0.

107. For what values of k does the equation $x^2 + (k+1)x + k + 4 = 0$ have real roots?

Use the discriminant. $b^2 - 4ac \geq 0$

Multiply out.

Tidy up.

Sketch the curve $y = k^2 - 2k - 15$
You need to factorise to see where it cuts the k axis.

What values of k make the y coordinates $\geq$ 0.

108. Find the largest and smallest value of p for which $p^2 - 2p \leq 3$

109. Find the set of values of x for which $7x - x^2 > 10$

Mixed questions 1

110. (a) Express $\sqrt{8}$ in the form $a\sqrt{b}$ where a and b are integers.

(b) Find x if $x^{\frac{1}{3}} = 8$

(c) Express $8^{-\frac{2}{3}}$ as a fraction

(d) Write 8^x as a power of 2

111. (i) Simplify $\left(2+\sqrt{3}\right)\left(2-\sqrt{3}\right)$

(ii) Express $\frac{3}{2+\sqrt{3}}$ in the form $a+b\sqrt{3}$ where a and b are integers.

112. Express $\sqrt{63}$ in the form $a\sqrt{b}$ where a and b are integers.

113. Simplify $\frac{x^{-1}}{\left(x^2\right)^{-1}}$

114. (a) Write the following as powers of 3 (i) $\sqrt{3}$ (ii) 27^x

(b) Hence solve the equation $27^x \times 3^{x-1} = \sqrt{3}$

115. Express $\frac{5}{\sqrt{7}+2}$ in the form $a+b\sqrt{7}$ where a and b are rational numbers.

116. If $2^x = 4\sqrt{2}$ find x.

117. Express p in terms of q if $3^p = 27^{q+1}$

118. Expand $\left(x^{\frac{1}{2}} - 3x^{\frac{3}{2}}\right)^2$

119. Given $x = 2^m$ and $y = 4^n$ find, in terms of m and n , $16xy$ as a power of 2

120. Simplify $4x^2 - (6 - x^2)$

121. Simplify $6x^3y + 3y^2 - 2y(x^2 - x^3 + y)$

122. Expand $(x^2 + 5x - 3)(3x^2 - 2)$

123. Expand $(x - 3)^2$

124. Factorise $10x^3 - 5x^2 + 20x$

125. Factorise $y^2 - 9$

126. Factorise $2x^3 - 32x$

127. Factorise $x^2 + 6x + 8$

128. Factorise $x^2 - 2x - 15$

129. Factorise $5x^2 + x - 18$

130. Factorise $5x^2 - 9x - 18$

131. Factorise $(x - 2)^2 - 5(x - 2)(x - 1)$

132. Simplify these fractions:

(a) $\frac{6x}{2}$ (b) $\frac{x^3y^2}{x^2y}$ (c) $\frac{9a^2}{6ab}$

133. Simplify these fractions:

(a) $\frac{x+2}{(x+2)(x+1)}$ (b) $\frac{x^2+3x}{x^2-2x-15}$ (c) $\frac{x^2-1}{x^2-7x+6}$

134. Work out:

(a) $\frac{5}{x} + \frac{3}{x}$ (b) $\frac{x}{2} - \frac{1}{y}$ (c) $\frac{3}{x} + \frac{2}{y}$

(d) $\frac{2}{x-1} + \frac{3}{2x}$ (e) $\frac{x}{3} + \frac{2}{x}$ (f) $\frac{1}{x+1} - \frac{2}{x-1}$

135. Write $x^2 + 6x + 8$ in completed square form. Hence solve the equation $x^2 + 6x + 8 = 0$

136. Complete the square for $3x^2 - 24x + 24$

137. Solve $x^2 - 5x = 0$

138. Solve $x^2 - 5 = 0$

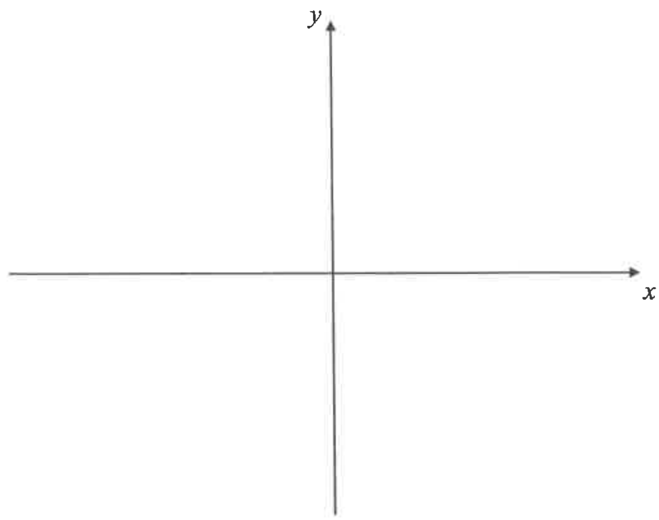
139. Solve $x^2 - x - 12 = 0$

140. Solve $x^2 = 7 - 2x$ by completing the square. Give the exact answers.

141. Solve $x^2 - 2x - 1 = 0$ giving the roots in surd form.

142. Solve $(2x-5)^2 = 2x$ giving solutions in simplified surd form.

143. Factorise and complete the square on x^2+x-6 Sketch the curve of $y=x^2+x-6$
Label the intercepts with the axes and the vertex clearly.



144. Write $2x^2+12x+16$ in the form $a(x+b)^2+c$

145. Solve the inequality $3x-2x^2 \le 0$

146. Find both solutions of the equation $x+3=\dfrac{10}{x}$

147. Find the values of p and q for which

$$x^2 - 6x + p = (x-q)^2 + 4$$

148. Solve the pair of equations

$$y - x = 4$$
$$y^2 - 5x^2 = 20$$

Comment on your solution.

149. By substituting $t = x^{\frac{1}{2}}$ find the values of x for which $x - 5x^{\frac{1}{2}} + 6 = 0$

150. Find the range(s) of values of x that satisfy the inequality $x^2 > 2x + 8$

151. Rearrange $y = \dfrac{x+1}{x-1}$ to find x in terms of y

COORDINATE GEOMETRY

When given coordinates it is usually helpful to sketch a grid with the approximate position of the points.
From your sketch you will be able to find:

- The gradient of lines between two given points.
- The midpoint of a line between two given points.
  The midpoint of a line joining (x_1, y_1) and (x_2, y_2) is $\boxed{\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)}$

Covered at CSE

- The distance between two given points by using Pythagoras' Theorem.
  The distance between (x_1, y_1) and (x_2, y_2) is $\boxed{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}}$

You may not have covered this specifically for your course, but it is useful to know. It is covered in the vectors topic.

The gradient of a straight line:

The gradient of a line m, through (x_1, y_1) and (x_2, y_2) is:

$$\boxed{m = \frac{y_2 - y_1}{x_2 - x_1}}$$

(Note the positions of y_2, x_2 and y_1, x_1. They must not be mixed up!)

You can easily see how to work out the results of these first three equations by drawing a sketch showing the approximate positions of the points. You would not have to use the equations.

Parallel lines have the same gradient.

If two perpendicular lines have gradients m_1 and m_2 then $\boxed{m_1 m_2 = -1 \text{ or } m_2 = \frac{-1}{m_1}}$

152. For a line joining the points $(-1, 5)$ and $(3, 1)$ Complete: Gradient of line = Gradient of a perpendicular line = Midpoint of line = Length of line =	*A sketch helps.* *Use the formulas or sketch to answer these.* *Leave in surd form. Remember:* *$\sqrt{12} = \sqrt{(4 \times 3)} = \sqrt{4} \times \sqrt{3} = 2\sqrt{3}$* *$\sqrt{45} = \sqrt{(9 \times 5)} = \sqrt{9} \times \sqrt{5} = 3\sqrt{5}$ etc.*

153. Find the distance between the points $(1, -1)$ and $(5, -10)$ giving your answer to 1 decimal place.

The equation of a straight line:

$\boxed{y = mx + c}$ Where m is gradient and the line cuts the y axis at $(0, c)$, the y axis intercept.

Another commonly used form is $\boxed{ax + by + c = 0}$

Straight line equations given in other forms can be rearranged into these forms.

154. Write $5x + 2y = 3$ in the form $y = mx + c$	155. Write $3y - 6x + 4 = 0$ in the form $y = mx + c$
156. Write $y = \frac{2x}{3} + 5$ in the form $ax + by + c = 0$	157. Write $y = \frac{5}{3} - \frac{3x}{2}$ in the form $ax + by + c = 0$

Drawing a line, given its equation:

There are several standard forms for the equation of a straight line.

(i) $\boldsymbol{x = a}$ A line parallel to the y axis
e.g. $x = 3$ passes through $(3, -2), (3, 0), (3, 7)$...
in fact all the points with x coordinate 3.

(ii) $\boldsymbol{y = b}$ A line parallel to the x axis
e.g. $y = -1$ passes through $(8, -1), (0, -1), (-4, -1)$...
in fact all the points with y coordinate -1.

(iii) $\boldsymbol{y = mx}$ A line through the origin with gradient m.
e.g. $y = -2x$

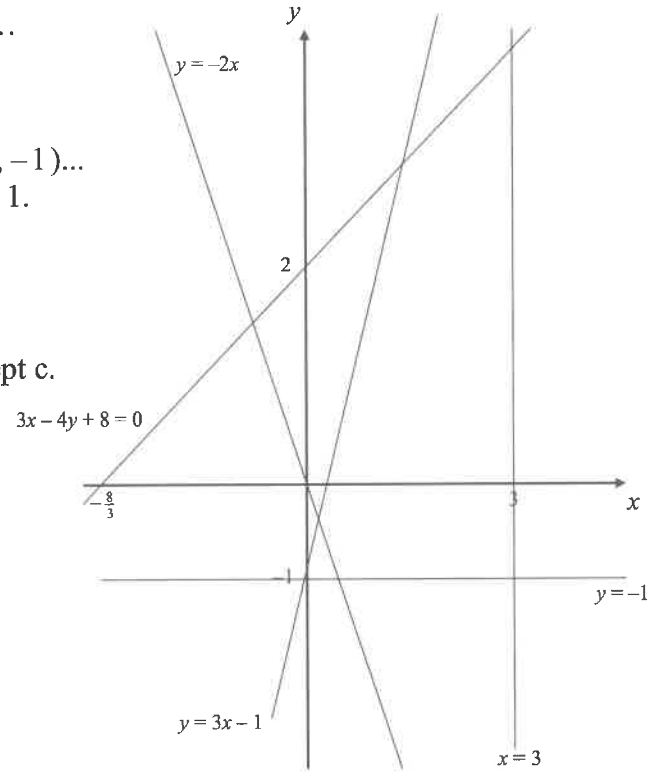
(iv) $\boldsymbol{y = mx + c}$ A line with gradient m and y intercept c.
e.g. $y = 3x - 1$

(v) $\boldsymbol{ax + by + c = 0}$ To draw this, either rearrange into $y = mx + c$ or substitute in $x = 0$ and $y = 0$ to find the points of intersection with the axes.

e.g. $3x - 4y + 8 = 0$

$x = 0 \Rightarrow -4y + 8 = 0 \Rightarrow y = 2$

$y = 0 \Rightarrow 3x + 8 = 0 \Rightarrow x = -\frac{8}{3}$



Sketching straight line graphs:

A sketch of a straight line graph needs to show the slope of the graph (+ve or –ve) and the intersection with the axes. You do not need to put scales on the axes.

158. Sketch the graph $2y - 3x + 6 = 0$

Complete:

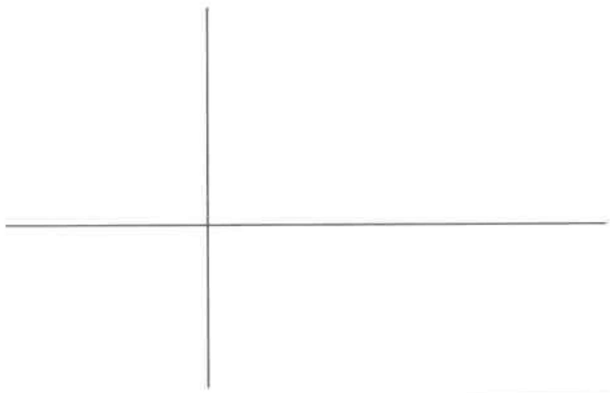
$x = 0 \quad y =$

$y = 0 \quad x =$

Find where the graph cuts the axes.

$y =$

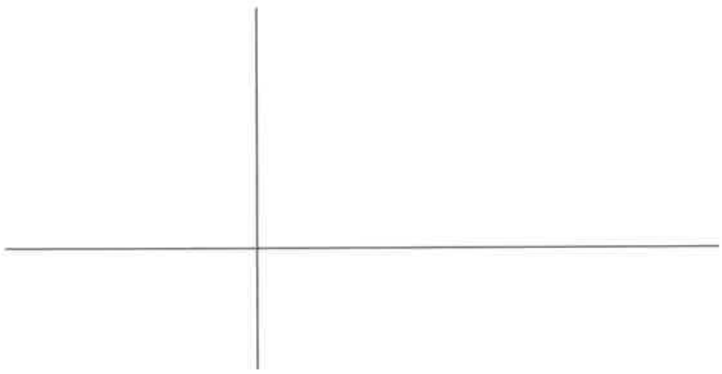
Rearrange to find the gradient. (Not needed but gives a check on your points.)



Sketch graph. (You do not need to put in scales for a sketch but label intercepts on x and y axes.)

Note: Always use a ruler for straight lines when drawing a sketch.

159. On the graph below sketch and label $y = 5$, $x = -2$, $y = x$, $y = 2x + 3$



Finding the equation of a line:
Given the gradient, m, and a point (x_1, y_1)

$$y - y_1 = m(x - x_1)$$

160. Find the equation of the line with gradient 3 that passes through the point $(2, 5)$ Complete: $y - 5 = 3(\quad\quad)$ $y - 5 =$ $y =$	*Multiply out the bracket.* (+5) *Alternative: Use $y = mx + c$. You know m. $\therefore y = 3x + c$ Substitute $x=2$ and $y=5$ to find c. Then write down the equation.*

Given two points (x_1, y_1) and (x_2, y_2).

Use $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ and $y - y_1 = m(x - x_1)$

OR

Use $\dfrac{y - y_1}{x - x_1} = \dfrac{y_2 - y_1}{x_2 - x_1}$

Note: The perpendicular bisector of a line:
- is perpendicular to the line and therefore has gradient (–1/gradient of line).
- divides the line in half and therefore the midpoint of the line is on the perpendicular bisector.

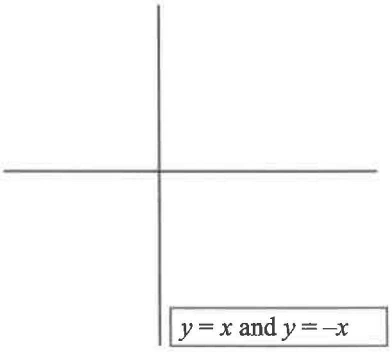
161. Find the equation of the line through $(-2, -3)$ and $(1, 5)$	*A sketch showing the relative positions of the points will help.* *Use the equation/s above to write down the equation of the line.* *Multiply out brackets and collect like terms and tidy up by multiplying* by 3 *OR* *Multiply both sides* by 3, *multiply out brackets and collect like terms.*
162. Find the equation of the line through $(2, -3)$ and parallel to the line $x - 2y + 5 = 0$	*Find the gradient of the given line by rearranging into the form y = ...* *Use the formula and tidy up.*

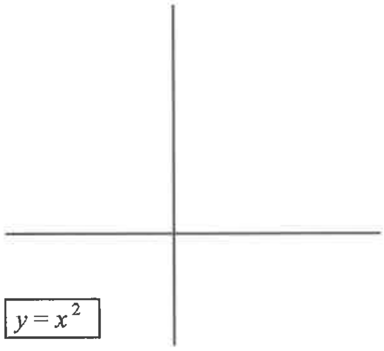
163. Find the equation of the line that passes through the point $(5, -3)$ and is perpendicular to the line $y = 2x - 1$	164. Find the equation of the line that passes through the point $(-1, -2)$ and is perpendicular to the line that passes through the points $(-1, 7)$ and $(8, -2)$

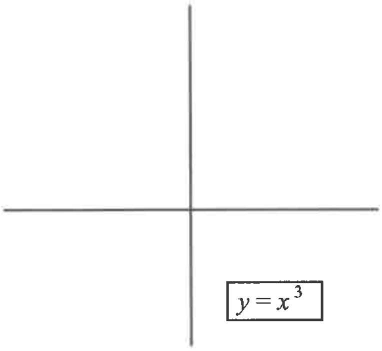
Intersection of two lines:
Two non-parallel lines will intersect at a point.
To find the point of intersection - solve the equations of the lines as simultaneous equations.
At the point of intersection the coordinates satisfy (fit) both equations simultaneously.

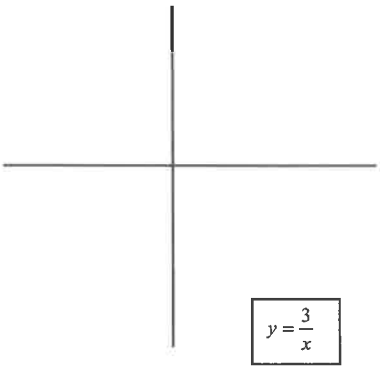
165. Find the intersection of the lines $y = 3x - 4$ and $4y + 2x = 7$	*Substitute $y = 3x - 4$ into second equation.* *Multiply out the bracket.* *Collect like terms.* *Find x.* *Substitute into $y = 3x - 4$ to find y.*

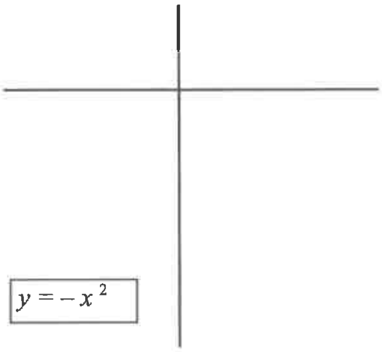
Graphs:
You should know the shape of certain curves. For each of the following sketch the curves marking in the important points and state any asymptotes.

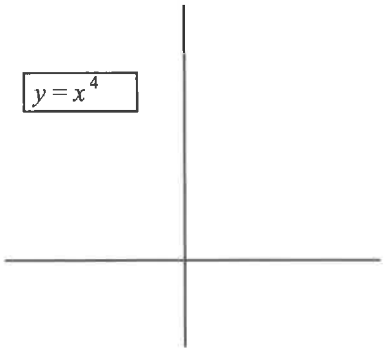


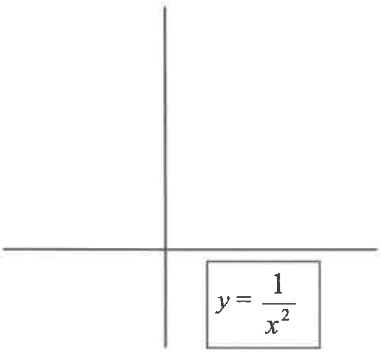


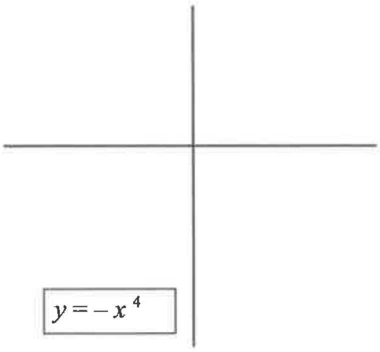


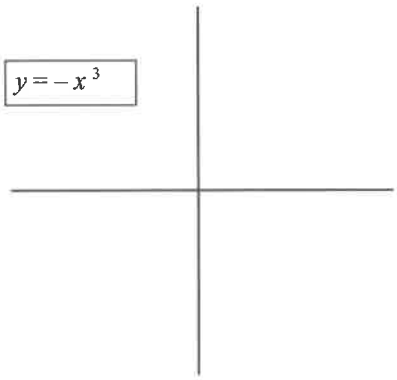












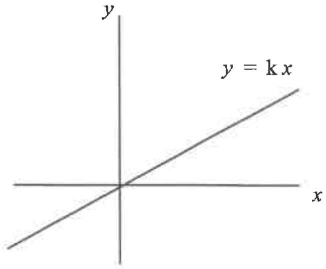
Check these by using a graphics calculator or computer.

Direct proportion:

y is proportional to x
y is directly proportional to x
y varies as x
y varies directly as x

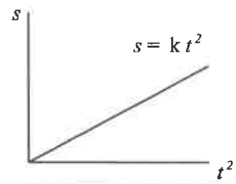
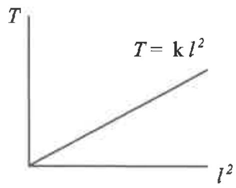
You write $y \propto x$

These all mean the same.
As one increases the other increases.
If x is doubled so is y.
If x is multiplied by 5 so is y.
etc.



Then $y = \mathrm{k}x$ k is called the constant of proportionality.

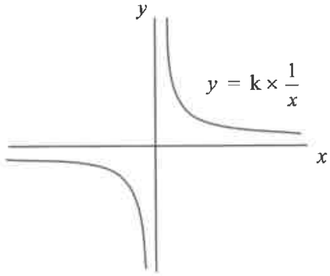
A graph showing direct proportion, will be a straight line graph through the origin, with gradient k.
If x can only be positive, for example a measured distance, then just the top right quadrant of the graph is valid.

166. The distance travelled by a car, d (metres), moving at constant speed, is directly proportional to the time taken, t (seconds). Given d = 15 when t = 4, Find: (i) d when t = 12 (ii) t when d = 60 Complete: d $\propto$ d = = k = so d = (i) (ii)	*Write with the proportionality sign.* *Write with the = and k.* *Substitute for d and t to find k.* ***Write the formula** with this value for k.* *(i) Use t = 12 to find d.* *(ii) Use d = 60 to find t.*
167. The distance travelled by a body under free fall, is modelled so that the distance travelled, s, is directly proportional to the time squared. If the distance travelled is 705.6 metres after 12 seconds, find the time taken to travel 122.5 metres.	*Write with the proportionality sign.* *(If t^2 is doubled so is s, etc.)* *Write with the = and k.* *Substitute for s and t to find k and **write the formula** with this value for k.* *Use s = 122.5 to find t.* *The graph of **s** against $\boldsymbol{t^2}$ would look like this.* 
168. The time period of a simple pendulum, is modelled so that the time period varies directly as the square root of the length of the pendulum. When the length of the pendulum is 1.5m, the time period is 2.45 seconds. Find the time period if the length of the pendulum is 2 metres. Give your answer to 3 significant figures.	*Write with the proportionality sign.* *Use T for time period and l for the length of the pendulum.* *(If $\sqrt{l}$ is doubled so is T, etc.)* *Write with the = and k.* *Substitute for T and l to find k and **write the formula** with this value for k.* *Use l = 2 to find T.* *The graph of **T** against $\boldsymbol{l^2}$ would look like this.* 

Inverse proportion:

y is inversely proportional to x

Here, as one increases the other decreases. If x is doubled y is halved. If x is multiplied by five, y is divided by five. etc.



You write $\quad y \propto \dfrac{1}{x}$

or $\quad y = \mathrm{k} \times \dfrac{1}{x} \qquad$ k is called the constant of proportionality.

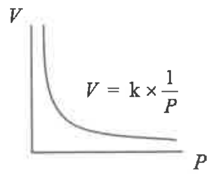
A graph showing inverse proportion will be a reciprocal graph.
If x can only be positive, for example a measured distance, then just the top right quadrant of the graph is valid.

169. p is inversely proportional to q and $\mathrm{p} = 2.5$ when $\mathrm{q} = 2$
Find p when $\mathrm{q} = 25$

Write with the proportionality sign.
Write with the = and k.
Substitute for p and q to find k and write the formula with this value for k.
Use q = 25 to find p.

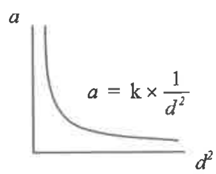
170. Boyle's law states that the volume of a given amount of gas, held at constant temperature, varies inversely with the applied pressure. A sealed box contains $5 \times 10^{-5}\,\mathrm{m}^3$ of air at pressure of $1.2 \times 10^5\,\mathrm{N/m}^2$. The box is squashed until the volume of trapped air is halved. If there is no change in temperature, what is the new pressure of the gas?

Write with the proportionality sign.
(Use V and P.)
Write with the = and k.
Substitute for V and P to find k and write the formula with this value for k.
Use V = 0.05 to find V.
The graph of ***V*** *against* ***P*** *would look like this.*



171. The acceleration due to gravity of an object changes with altitude. The gravitational acceleration is inversely proportional to the square of the distance from the centre of Earth. Modelling the distance to the centre of the Earth as 6371 km and the gravitational acceleration as $9.8\,\mathrm{ms}^{-2}$ at the surface of the Earth, find the gravitational acceleration 1000 km above the surface of the Earth.

Write with the proportionality sign.
(Use a and d^2.)
Write with the = and k.
Substitute for a and d to find k and write the formula with this value for k.
Use d = 1000 + 6371 to find a.
The graph of ***a*** *against* $\boldsymbol{d^2}$ *would look like this.*



Sketching polynomials:

Cubics

To find where a curve cuts y axis put $x = 0$

To find where a curve cuts/touches the x axis put $y = 0$

Given $y = (x+1)(x-2)(x-3)$

Let $x = 0$, $y = 6$

Let $y = 0$, $0 = (x+1)(x-2)(x-3)$

Three brackets multiply to give 0 so one must be 0.

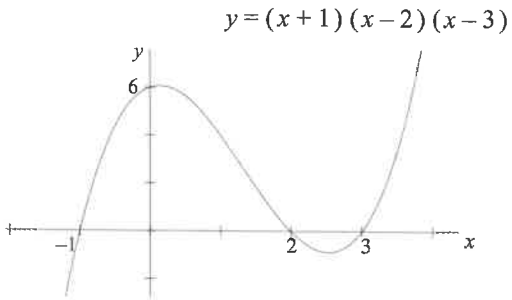
Either $x + 1 = 0$ or $x - 2 = 0$ or $x - 3 = 0$

$\therefore \quad x = -1$, 2 or 3

The curve is $y = (x+1)(x-2)(x-3)$
$= x^3 - 4x^2 + x + 6$

x^3 is dominant so the curve is like $y = x^3$ for large +ve and –ve values of x

A sketch of the curve:

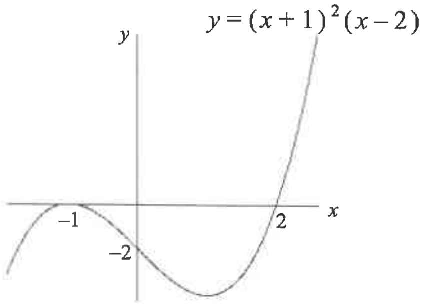


If there is a squared factor then the curve touches the x axis

e.g. $y = (x+1)^2(x-2)$

when $y = 0$ $x = 2$ or -1 twice

x^3 is dominant so the curve is like $y = x^3$ for large +ve and –ve values of x



Quartics

To find where a curve cuts y axis put $x = 0$

To find where a curve cuts/touches the x axis put $y = 0$

Given $y = (x+1)^2(x-1)(x-2)$

Let $x = 0$, $y = 2$

Let $y = 0$, $0 = (x+1)^2(x-1)(x-2)$

All brackets multiply to give 0 so one must be 0.

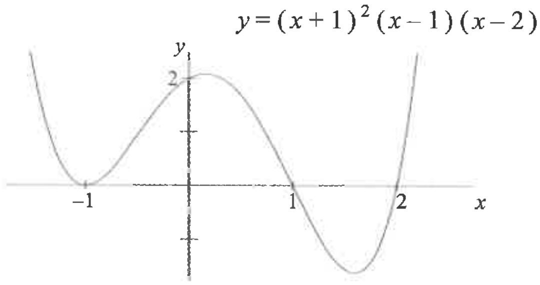
Either $x = 0$ or $x + 1 = 0$ or $x - 2 = 0$ or $x - 3 = 0$

$\therefore \quad x = -1$ twice, 1 or 2

The curve is $y = (x+1)^2(x-1)(x-2)$
$= x^4 - x^3 - 3x^2 + x + 2$

x^4 is dominant so the curve is like $y = x^4$ for large +ve and –ve values of x

A sketch of the curve:

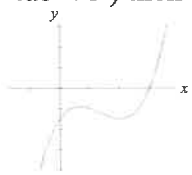


There is a squared factor so the curve touches the x axis at $x = -1$

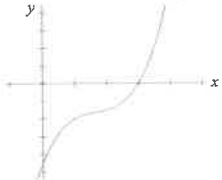
You should know:

- the basic shapes for $y = x^3$, $y = -x^3$, $y = x^4$ and $y = -x^4$
- the position of graph for large positive values of x and large negative values of x
  e.g. If $y = x^3 + 2x^2 - 5x - 6$ then for large +ve values of x the x^3 term is dominant and so the graph is similar to that for $y = x^3$ for large x.
  If $y = -x^3 + 2x^2 - 5x - 6$ then for large +ve values of x the $-x^3$ term is dominant and so the graph is similar to that for $y = -x^3$ for large x.
  Similarly for a quartic.
- how to find where the curve cuts the y axis (where $x = 0$).
- how to find where the curve cuts the x axis (where $y = 0$).
- how to find the stationary points by differentiation - covered later.

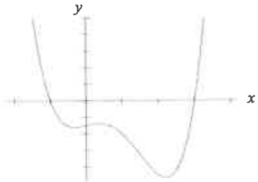
If there is a quadratic factor that does not factorise, or using the quadratic formula does not give real roots ($b^2 - 4ac < 0$) then the curve will not further cut or touch the x axis.



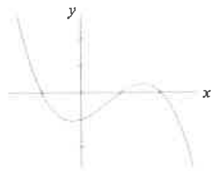
$y = (x-3)(x^2 - x + 1)$



$y = (x-3)(x^2 - 2x + 3)$



$y = (x-3)(x+1)(x^2 - x + 1)$



$y = (1-x)(x-2)(x+1) = -(x-1)(x-2)(x+1)$

Note: $(1-x) = -(x-1)$
so
$y = (1-x)(x-2)(x+1)$
$= -(x-1)(x-2)(x+1)$

In this section we are interested in where the curve cuts/touches the axes.
Do not find the stationary points in these questions.

172. Sketch these curves showing where the curves cut the axes:

(a) $y = (x+2)(x-1)(x-3)$ (b) $y = (x-1)^2 (x+3)$

173. Sketch these curves showing where the curves cut the axes:

(a) $y = x^2 (x-2)$ (b) $y = (2-x)(x-1)(x-4)$

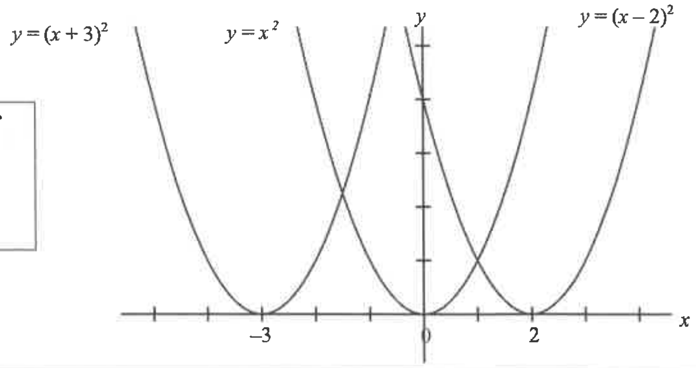
174. Sketch these curves showing where the curves cut the axes:

(a) $y = x^2 (x-2)(x+1)$ (b) $y = (x+1)(x-1)(x-2)(x+3)$

Transformations of graphs:

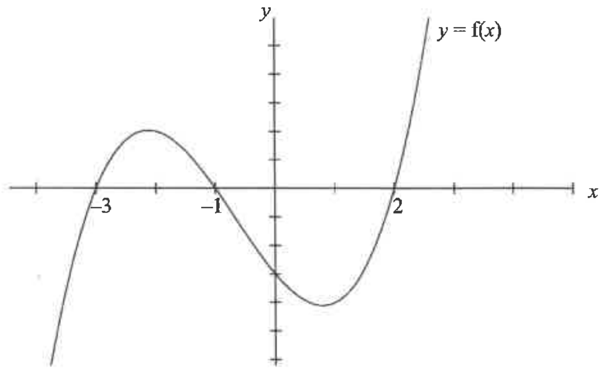
These graphs show:

A curve with equation $y = f(x+k)$ is a translation of $y = f(x)$ by vector $\begin{pmatrix} -k \\ 0 \end{pmatrix}$



175. Here is the graph $y = f(x)$ where
$f(x) = (x-2)(x+1)(x+3)$

On the same graph sketch $y = f(x-2)$
Label where the graph cuts the x axis.

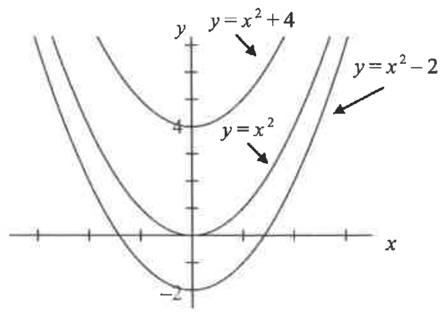


Write down the algebraic equation of $f(x-2)$
(Replace every x in $f(x)$ by $(x-2)$ and simplify.)

$f(x-2) =$

These graphs show:

A curve with equation $y = f(x) + k$ is a translation of $y = f(x)$ by vector $\begin{pmatrix} 0 \\ k \end{pmatrix}$

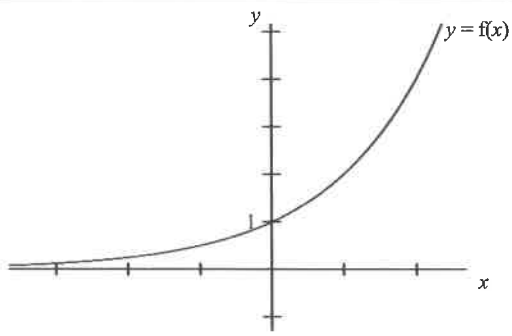


176. Here is the graph $y = f(x)$ where $f(x) = 2^x$

On the same graph sketch $y = f(x) + 1$

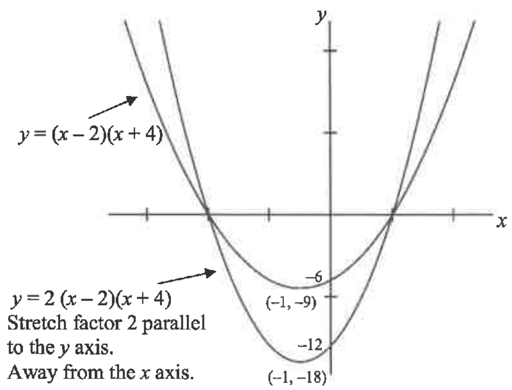
Write down the algebraic equation of $f(x) + 1$

$f(x) + 1 =$



These graphs show:

A curve with equation $y = af(x)$ is a stretch of factor a parallel to the y axis. (Away from the x axis.)



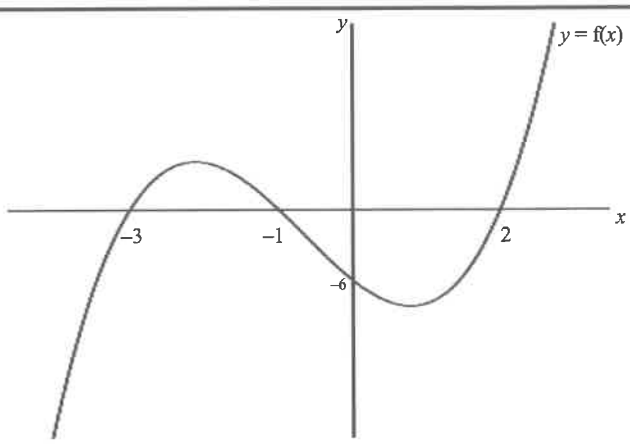
177. Here is the graph $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = (x-2)(x+1)(x+3)$$

On the same graph sketch $y = 2\,\mathrm{f}(x)$

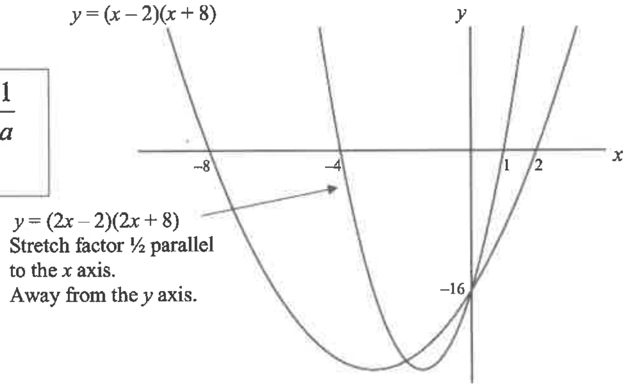
Write down the algebraic equation of $2\,\mathrm{f}(x)$
(Just multiply whole of $\mathrm{f}(x)$ by 2)

$2\,\mathrm{f}(x) =$



These graphs show:

A curve with equation $y = \mathrm{f}(ax)$ is a stretch of factor $\frac{1}{a}$ parallel to the x axis. (Away from the y axis.)



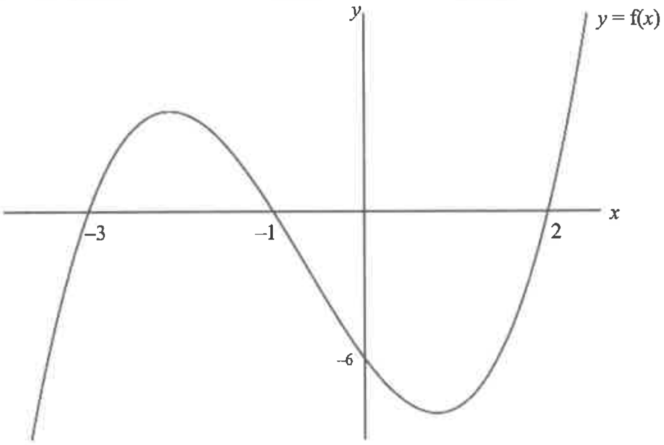
178. Here is the graph $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = (x-2)(x+1)(x+3)$$

On the same graph sketch $y = \mathrm{f}(2x)$
Label where the graph cuts the x axis.

Write down the algebraic equation of $\mathrm{f}(2x)$
(Replace every x in $\mathrm{f}(x)$ by $2x$)

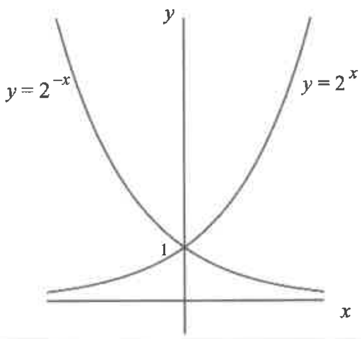
$\mathrm{f}(2x) =$



Useful to know this particular case.

These graphs show:

A curve with equation $y = \mathrm{f}(-x)$ is a reflection of $y = \mathrm{f}(x)$ in the y axis.

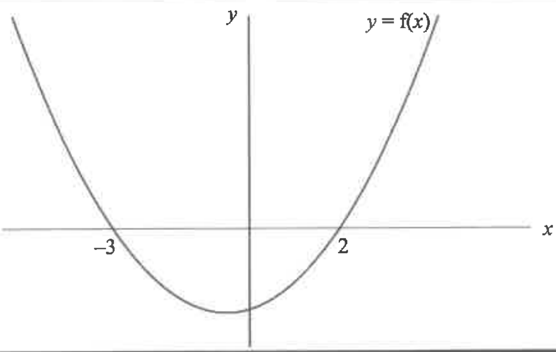


179. Here is the graph $y = \mathrm{f}(x)$ where $\mathrm{f}(x) = (x-2)(x+3)$
$= x^2 + x - 6$

On the same graph sketch $y = \mathrm{f}(-x)$
Label where the graph cuts the x axis.

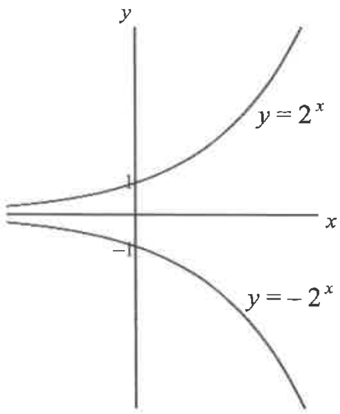
Write down the algebraic equation of $\mathrm{f}(-x)$
(Replace every x in $\mathrm{f}(x)$ by $(-x)$ and simplify.)

$\mathrm{f}(-x) =$



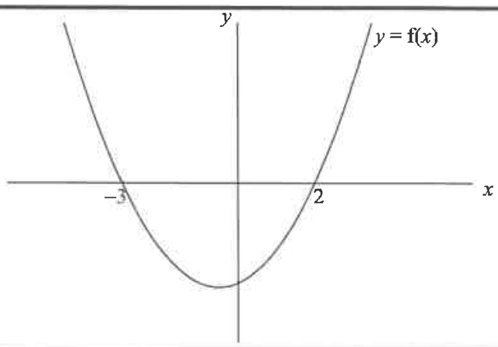
Useful to know this particular case.

These graphs show:



A curve with equation $y = -\mathrm{f}(x)$ is a reflection of $y = \mathrm{f}(x)$ in the x axis.

180. Here is the graph $y = \mathrm{f}(x)$ where $\mathrm{f}(x) = (x-2)(x+3)$
$= x^2 + x - 6$

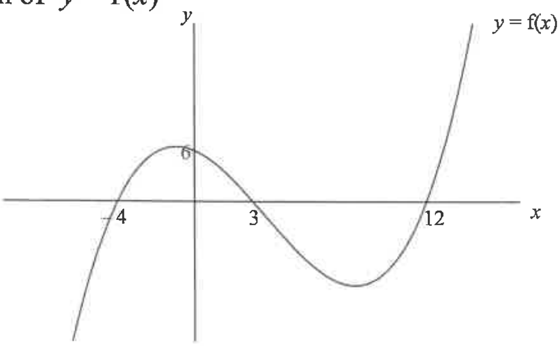


On the same graph sketch $y = -\mathrm{f}(x)$
Label where the graph cuts the x axis.

Write down the algebraic equation of $-\mathrm{f}(x)$

$-\mathrm{f}(x) =$

181. Here is a sketch of $y = \mathrm{f}(x)$



On separate diagrams sketch the following:
(a) $y = \mathrm{f}(3x)$ (b) $y = \mathrm{f}(-x)$ (c) $y = 3\mathrm{f}(x)$ (d) $y = \mathrm{f}(x+3)$

Label the important points. i.e. where the curve cuts the axes. Label your graphs.

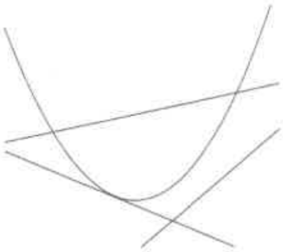
182.
(a) Fully describe a single geometrical translation, that maps the graph of $y = 2^x$ onto the graph of $y = 2^{x+1}$
(b) Fully describe a single geometrical stretch, that maps the graph of $y = 2^x$ onto the graph of $y = 2^{x+1}$

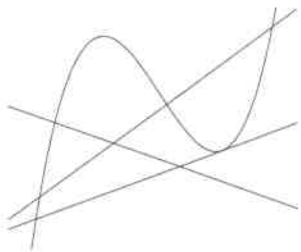
183. The graph of $y = x^2 - 3x + 5$ is translated by the vector $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$

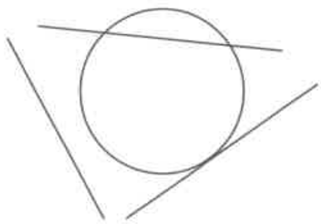
Find the equation of the translated graph, giving your answer in its simplest form.

Intersection of straight lines and curves:

To find the points of intersection, substitute for x or y from the straight line equation into the other equation and solve.







Straight lines and quadratic curves will:
- Intersect twice - where they intersect there will be two distinct roots (solutions) when solving equations to find x or y.
- Touch - where they touch there will be equal roots when solving equations to find x or y. The line is a tangent to the curve.
- Not meet - there will be no real roots when solving equations to find x or y.

Straight lines and cubics will:
- Intersect at three places - where they intersect there will be three distinct roots (solutions) when solving equations to find x or y.
- Intersect once and touch once - there will be three roots (solutions) when solving equations to find x or y. One double root (where the line touches) and another distinct root (where the line cuts).
- Intersect once - there will one linear factor and a quadratic factor that does not have any real roots.

Except $y = x^3$ (at $x = 0$) and transformations.

Straight lines and circles will:
- Intersect - where they intersect there will be distinct roots (solutions) when solving equations to find x or y.
- Touch - where they touch there will be equal roots when solving equations to find x or y. The line is a tangent to the curve.
- Not meet - there will be no real roots when solving equations to find x or y.

184. Find the coordinates of the points of intersection of the curve $y = x^2 - 1$ and the line $y + x = 5$ Sketch both curves.

Rearrange the linear equation to give y in terms of x

Equate.

Tidy up.

Solve the resulting equation by factorising, or using the quadratic formula.

Find the corresponding values of y.

Sketch. Label the points of intersection.

Mixed questions 2

185. Find the equation of the line through the points $(-2, 11)$ and $(4, 7)$

186. Find the equation of the perpendicular bisector of the line joining the points $(1, 7)$ and $(9, 3)$

187. The electrical resistance of a wire is inversely proportional to the square of the radius. Given that the resistance is 0.02 ohms when the radius is 0.1 cm, find the resistance when the radius is 0.3 cm. Give your answer to 1 significant figure.

188. Triangle ABC has vertices A $(-1, 2)$, B $(5, 2)$ and C $(4, 9)$. N is the foot of the perpendicular from B to AC. Find: (a) the equation of AC (b) the equation of BN.

189. Show that the line $y = 2x - 7$ touches the curve $y = x^2 - 6$

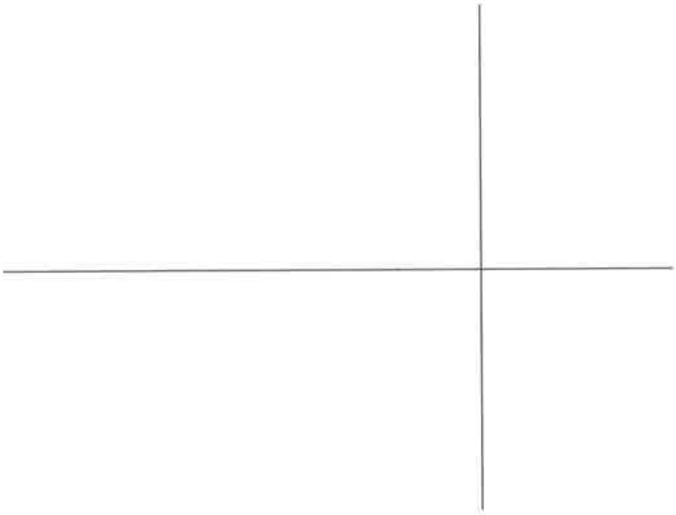
190. If $f(x) = x^2 + x - 2$

(i) Find where the curve $y = f(x)$ cuts the axes.

(ii) By completing the square find the minimum point of $y = f(x)$

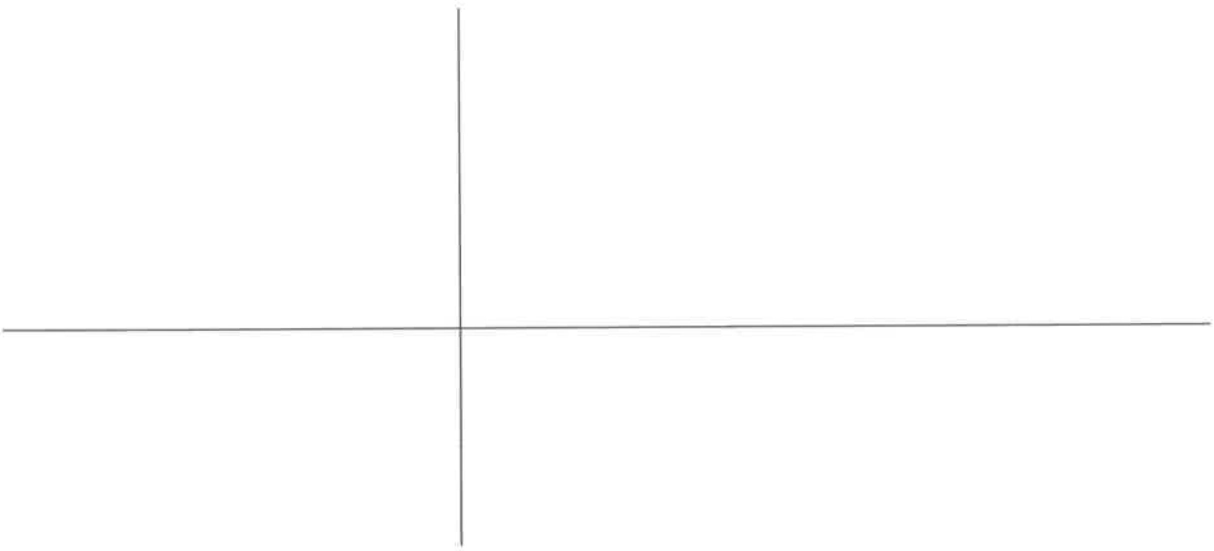
(iii) Sketch $y = f(x)$

(iv) On the same diagram sketch and label $y = 2f(x)$



191. The coordinates of A, B and C are $(3, -1)$, $(5, 3)$ and $(2, 2)$ respectively. Find the length AB, in the form $a\sqrt{b}$, and the mid-point of AB. Show that ACB is a right angle.

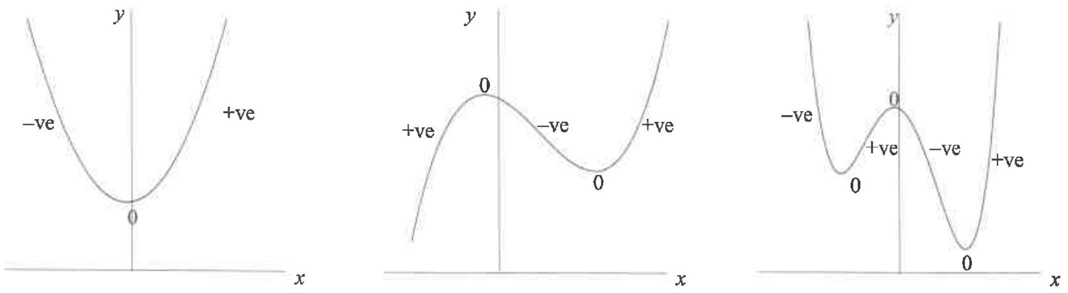
192. $f(x) = x(x-3)^2$ (i) Sketch $y = f(x)$ (ii) On the same diagram sketch $y = f(3x)$.



DIFFERENTIATION

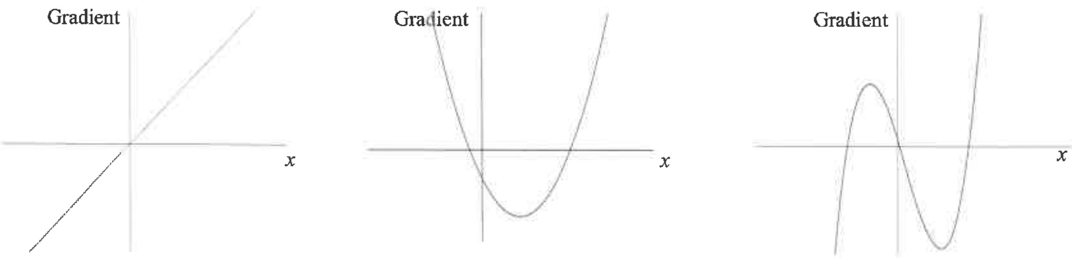
Finding the gradient function:

For each of the following graphs the gradient changes smoothly as you move along the curve as shown:



If you work out the gradient at various points on the above graphs and plot them against x, you would obtain the following:

Graphs of the gradients. These show how the gradient changes as x changes i.e. a function of x - **the gradient function.**



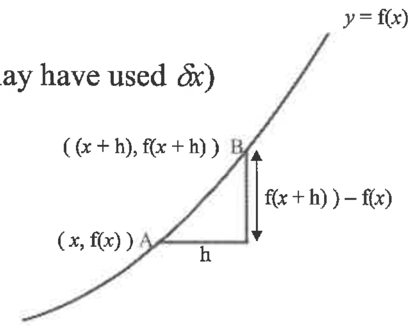
Finding the gradient function is called differentiation.

Differentiation from first principles:

Consider the graph of $y = f(x)$ and a general point on the curve $(x, f(x))$

The gradient of the curve at this point ≈ Gradient of AB for small h (you may have used δx)

If you make h small enough the gradient of AB will equal the gradient of the curve at A. This is called taking the limit as h tends to 0 and is written $\lim\limits_{h\to 0}$ (gradient of AB)



The gradient of the curve $y = f(x)$ is written $\dfrac{dy}{dx}$

and so $\quad \dfrac{dy}{dx} = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

Example: Differentiate x^2 from first principles.

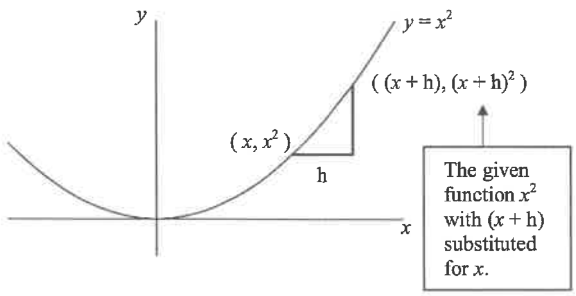
$$\frac{dy}{dx} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= \lim_{h\to 0} \frac{2xh + h^2}{h} = \lim_{h\to 0} \frac{2xh}{h} + \frac{h^2}{h}$$

$$= \lim_{h\to 0} 2x + h$$

$$= 2x$$



Example: Differentiate $2x^2 + 5x - 3$ from first principles.

A sketch of part of a curve will help you 'see' what you are doing.

Let $f(x) = 2x^2 + 5x - 3$

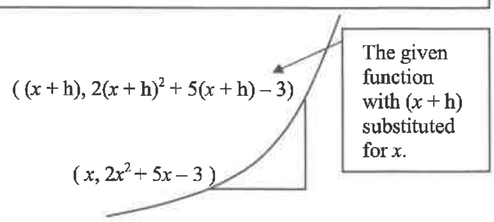
$$f(x+h) - f(x) = 2(x+h)^2 + 5(x+h) - 3 - (2x^2 + 5x - 3)$$

$$= 2x^2 + 4xh + 2h^2 + 5x + 5h - 3 - 2x^2 - 5x + 3$$

$$= 4xh + 2h^2 + 5h$$

$$\therefore \quad \frac{dy}{dx} = \lim_{h\to 0} \frac{4xh + 2h^2 + 5h}{h} = \lim_{h\to 0} 4x + 2h + 5$$

$$= 4x + 5$$



193. Differentiate x^3 from first principles.

194. Differentiate $\frac{1}{x}$ from first principles.

Differentiating using standard results:

Notation: Given $y = x^2$ you write $\frac{dy}{dx} = 2x$

Given $f(x) = x^2$ you write $f'(x) = 2x$

Given 'differentiate x^2' you write $\frac{d(x^2)}{dx} = 2x$ Or you could write ' Let $y = x^2$ then $\frac{dy}{dx} = 2x$ '

$$y = kx^n \Rightarrow \frac{dy}{dx} = knx^{n-1}$$

Where k is a constant (i.e. 2 or 3 or –2 or – 1 or ½ or......... etc).

$$y = c \Rightarrow \frac{dy}{dx} = 0$$

Where c is a constant (i.e. 2 or 3 or –2 or – 1 or ½ or......... etc).

$$y = f(x) + g(x) \Rightarrow \frac{dy}{dx} = f'(x) + g'(x)$$

Differentiate each term.

Note: For $y = kx$ $\frac{dy}{dx} = k$

So if $y = 5x$ $\frac{dy}{dx} = 5$ etc.

195. Complete: (a) $y = x^3 \Rightarrow \frac{dy}{dx} =$ (b) $y = 6x^4 \Rightarrow \frac{dy}{dx} =$

(c) $f(x) = 5x^2 \Rightarrow f'(x) =$ (d) $y = 3x^2 + 2x - 4 \Rightarrow \frac{dy}{dx} =$

If you are asked to find the derivative of a function, or the derived function, or the gradient function, then just differentiate the function you are given.
To find the gradient of a curve at a point - differentiate and then substitute the x coordinate of the point.

196. (a) Find the gradient function for $y = 6x^5$	*This means differentiate!*
Complete: $\frac{dy}{dx} =$	
(b) Find the gradient of the curve $y = x^4 - 3x^2 + 2x - 5$ at the point $(2, 3)$	
Complete: $\frac{dy}{dx} =$	*Differentiate.*
When $x = 2$, $\frac{dy}{dx} =$	*Substitute. (Given in question.)*

197. Find the point on the curve $y = x^2 - 4x$ where the gradient equals 8. Complete: $\frac{dy}{dx} =$ $8 =$ $x =$ (,)	*Differentiate to find the gradient.* *Put equal to 8.* *Solve to find x.* *Use $y = x^2 - 4x$ to find y coordinate and write down the point where the gradient equals 8.*

Rates of change:

$\frac{dy}{dx}$ is the rate at which y changes as x changes. $\frac{dy}{dx} = 3$ means y is changing 3 times as fast as x (if x changes by 5, y will change by 15).

The rate of change of p with change in z is $\frac{dp}{dz}$ or the rate of change of p with respect to z. i.e. differentiate p with respect to z. e.g. If $p = 2z^3 + 5z - 4$ then $\frac{dp}{dz} = 6z^2 + 5$

198. The velocity of a particle is given by $v = 3t^2 - 2t$ Find the acceleration (the rate of change in velocity) when $t = 2$	*The rate of change of velocity with respect to time. i.e. differentiate v with respect to t.*
199. The side length of a cube is increasing. Find the rate at which the volume is changing with change in the side length.	*Write V =* *The rate of change of V with respect to side length. i.e. differentiate V with respect to side length.*

For any ***rational*** number n, $y = \mathrm{k}x^n, \quad \frac{dy}{dx} = \mathrm{k}\,n\,x^{n-1}$

Rational means that n can be a positive, or a negative integer, or a fraction and the rule still works!

Examples: $y = x^{\frac{1}{2}} \Rightarrow \frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$ $\quad y = x^{-3} \Rightarrow \frac{dy}{dx} = -3x^{-4}$	*Take care when you are subtracting 1 from the power. Fractional and negative powers are tricky.* *$-5-1=-6$ not -4 and $\frac{2}{3}-1=-\frac{1}{3}$ whilst $-\frac{2}{3}-1=-\frac{5}{3}$ etc.*
Example: Find the gradient function for $\frac{7}{3\sqrt{x^3}}$ If we write $y = \frac{7}{3}x^{-\frac{3}{2}}$ and use the power rule $\frac{dy}{dx} = \frac{7}{3}\times(-\frac{3}{2})\ x^{-\frac{5}{2}} = -\frac{7}{2}\ x^{-\frac{5}{2}}$	*Any constant multipliers at the front of the expression are unaffected by the differentiation. DO NOT bring the 3 to the top.* *Here the $\frac{7}{3}$ remains unchanged.* $\sqrt{x} = x^{\frac{1}{2}}$ *and* $\sqrt{x^3} = x^{\frac{3}{2}}$ *and* $\frac{1}{\sqrt{x^3}} = x^{-\frac{3}{2}}$

Complete the following:

200. $y = \sqrt{x^3} =$ $\quad \frac{dy}{dx} =$	*Remember:* $\sqrt{x} = x^{\frac{1}{2}}$ *etc.*
201. $y = \frac{1}{x^2} =$ $\quad \frac{dy}{dx} =$	*Rewrite the function using a negative power.*
202. $f(x) = \frac{2}{\sqrt{x}} =$ $\quad f'(x) =$	*Rewrite the function using a negative power.*

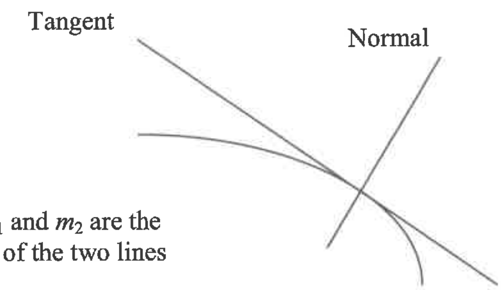
203. $y = \frac{1}{x} - \sqrt{x} + 3x^2 =$ $\frac{dy}{dx} =$	*Write as powers of x.* *Differentiate each term.*
204. $y = \sqrt[3]{x^2} =$ $\frac{dy}{dx} =$	*Write as powers of x. Cube root - power ⅓ .* *Differentiate.*
205. $y = \frac{3}{2x} =$ $\frac{dy}{dx} =$	*Write as powers of x.* *Careful - Only put the x to the top as a negative power.* *Differentiate.*
206. $y = \frac{2x^3 - 3x^2}{x} =$ $\frac{dy}{dx} =$	*First write as two separate fractions and simplify or divide numerator and denominator by x. Simplify.* *Differentiate.*
207. $y = \sqrt{x}(x+2) =$ $\frac{dy}{dx} =$	*Write the* $\sqrt{x}$ *as a power of x. Then multiply each term in the bracket by it.* *Differentiate.*
208. $f(x) = x(\sqrt{x} + 2) =$ $f'(x) =$	*Write the* $\sqrt{x}$ *as a power of x.* *Then multiply each term in the bracket by the x.* *Differentiate.*
209. The population, P, of a colony of birds on an island is modelled by the function $P = 100\sqrt{t} + 1000$ where t is the time in years from the initial census of birds on the island. Find the rate at which the population is changing after 4 years.	*The rate of change of P with respect to time.* *i.e. differentiate P with respect to t.*

Tangents and normals:

You need to know:

- How to differentiate.
- How to find the equation of a straight line.
- For perpendicular lines $m_2 = \frac{-1}{m_1}$

Where m_1 and m_2 are the gradients of the two lines



210. Find the equation of the tangent and the normal to the curve $y = 2x^3 - 5x^2 + 2x + 5$ at the point $(1, 4)$

Complete: $\frac{dy}{dx} =$

Differentiate.

When $x = 1$ $\frac{dy}{dx} =$

$\therefore$ Gradient of tangent, m =

Find gradient of curve when $x = 1$ This equals the gradient of the tangent. i.e. m

Use $(y - y_1) = m(x - x_1)$ or $y = mx + c$ to find the equation of the tangent (which is a straight line).

$\therefore$ Equation of tangent is

Gradient of normal

Use $m_2 = -1/m_1$

Use $(y - y_1) = m(x - x_1)$ or $y = mx + c$ to find the equation of the normal (which is a straight line).

$\therefore$ Equation of normal is

211. Find the equation of the tangent to the curve $y = 2x^2 - 2x + 5$ at the point where $x = -1$

212. Find point where the curve $y = x^2 - x - 6$ has gradient equal to 8. Work out the equation of the normal at this point.

INTEGRATION

Standard results:

$$\int a x^n dx = \frac{a x^{n+1}}{n+1} + c \qquad n \neq -1$$

Example: $\int 2x^3 dx = \frac{2x^4}{4} + c = \frac{x^4}{2} + c$

$$\int k\, dx = k x + c$$

Example: $\int 3\, dx = 3x + c$

$$\int f(x) + g(x)\, dx = \int f(x) dx + \int g(x) dx$$

Example:
$\int 2x^2 + 3x\, dx = \frac{2x^3}{3} + \frac{3x^2}{2} + c$

Given $\frac{dy}{dx}$, integrate to find y.

Given a point on the curve, substitute the x and y values of the point to find the constant. Write the solution with the value of the constant.

Examples:

Find y if $\frac{dy}{dx} = 3x^2 + 2x$

$y = \int 3x^2 + 2x\, dx = x^3 + x^2 + c$

Find y if $\frac{dy}{dx} = 2x - 3$ and the curve passes through (1, 2)

$y = \int 2x - 3\, dx = x^2 - 3x + c$

$x = 1 \;\; y = 4 \;\; \therefore \;\; 4 = 1 - 3 + c \Rightarrow c = 6$

so $\;\; y = x^2 - 3x + 6$

213. $\int 5x^4 dx$	*Don't forget the constant of integration. Tidy up.*
214. $\int 3x^2 - 2x + 4\, dx$	*Integrate each term.*
215. $\int 4x(x-2)\, dx$	*Multiply out then integrate each term.*
216. $\int 5 - x^3\, dx$	*Integrate each term.*
217. $\int (2x-3)(x+2)\, dx$	*Multiply out then integrate each term.*
218. $\int \frac{x^3 - 4x}{x}\, dx$	*First, divide top and bottom by x.*
219. $\int \sqrt{x} + \frac{3}{x^2} - 3x\, dx$	*Rewrite as powers of x. Then integrate each term.*
220. If $\frac{dy}{dx} = 3x^3 + x$ find y	*Integrate.*
221. Find y if $\frac{dy}{dx} = \sqrt{x^3} + \frac{2}{x^3} - 3$	*Rewrite as powers of x. Then integrate each term.*

Mixed questions 3

222. Find the gradient of $y = x^3 - 3x^2 + 7$ where $x = 2$

223. The equation of a curve is $y = x^2 - x - 12$
Find: (a) the gradient where the curve cuts the y axis
(b) the gradient at each point where the curve cuts the x axis.

224. Find the coordinates of the points on the curve $y = x^3 - 12x + 9$ where the gradient is 15.

225. Find $\dfrac{dy}{dx}$ when $y = 2x + 3\sqrt{x} - \frac{4}{x}$

226. A curve has equation $y = 2x^2 + 5x - 4$
Find: (a) the gradient of the tangent where $x = 2$
(b) the equation of this tangent
(c) the gradient of the normal where $x = 1$
(d) the equation of this normal.

227. Find the equation of the tangents to the curve $y = x^2 - 5x + 4$ at the points where the curve crosses the x axis. Find the coordinates of the point where these tangents meet.

228. A curve has equation $y = (x+4)(x-1)(x-3)$ Find:

(a) the equation of the tangent at $(2, -6)$
(b) the coordinates of the point where this tangent meets the x axis
(c) the equation of the normal at the point where $x = 1$
(d) the coordinates of the point where this normal and the tangent meet.

229. Find $\frac{dy}{dx}$ for each of the following:

(a) $y = x^3 - 5x + 4$ (b) $y = x(x+4)(x-5)$ (c) $y = 2\sqrt{x^3}$ (d) $y = \frac{3}{x^2} + \frac{7}{2\sqrt{x}}$ (e) $y = \frac{2x^2 - 5x}{\sqrt{x}}$

230. Integrate $3x^3 - 4x - 5$
231. $\int x^2 + 3x - 4\,dx$
232. $\int 4\sqrt{x} - \frac{2}{x^2}\,dx$
233. $\int x^2(x-6)\,dx$
234. $\int \frac{x^3+3x}{x}\,dx$
235. If $\frac{dy}{dx} = 3x^2 + 4x$ find y given that $y = 7$ when $x = 1$
236. $\int 1 - \sqrt[3]{t}\,dt$
237. Integrate $(3x+1)^2$
238. Find y if $\frac{dy}{dx} = 8x + 1$ and the curve passes through $(2, 5)$
239. Find y if $\frac{dy}{dx} = \frac{4}{x^2}$ and the curve passes through $(1, -4)$. Sketch the curve.

Mixed Questions 4

240. If $f(x) = x^2$ sketch the graph $y = f(x)$
(a) On the same diagram sketch and label the curves $y = f(x+3)$ and $y = -f(x+3)$
(b) Write down expressions for $f(x+3)$ and $-f(x+3)$ in terms of x.

241. Find the equation of the line through $(-2, 5)$ which is parallel to the line $y = 6 - 4x$

242. Express $\dfrac{(\sqrt{2}-1)^2}{\sqrt{2}+1}$ in the form $a + b\sqrt{2}$ where a and b are integers.

243. Given $8^x = 4^{x-1}$ find the value of x

244. Find $\int \frac{1}{3}\sqrt{x^3} - \frac{2}{\sqrt{x}}\, dx$

245. The volume of a container is directly proportional to the square of its height. The volume is 279 cm^3 when the height is 8 cm. What is the height if the volume is doubled? Give your answer to 3 significant figures.

246. Given that $f(x) = x^2 + kx + 4$ and that the equation $f(x) = 0$ has two real distinct roots, find the set of values that k can take.

247. Find the points of intersection of the curve $y = x^2 - 6$ and the line $y = x + 6$

248. Sketch the graph $y = x^2 + 3x - 10$ labelling all intersections with the axes.
State the range of values of x for which $x^2 + 3x - 10 < 0$

249. Find the equation of the line through $(1, -3)$ which is perpendicular to the line $y - 2x + 5 = 0$

250. Evaluate $8^{-\frac{1}{3}} + 27^{\frac{2}{3}}$

251. Find $\frac{dy}{dx}$ in each of the following (i) $y = \sqrt{x} + \frac{1}{\sqrt{x}}$ (ii) $y = x(x^3 - 2x)$ (iii) $y = \frac{3x^3+2x^2}{x}$

252. Calculate the discriminant of $2x^2 + 3x + 4$ and hence state the number of real roots of the equation $2x^2 + 3x + 4 = 0$

253. Complete the square on $x^2 - 6x + 5$ and hence state the coordinates of the vertex of the graph $y = x^2 - 6x + 5$

254. Given $\left(2-\sqrt{5}\right)\left(3-\sqrt{5}\right)=a+b\sqrt{5}$ where a and b are integers (a) find the values of a and b.

Given $\frac{2-\sqrt{5}}{3+\sqrt{5}} = c + d\sqrt{5}$, where c and d are rational numbers (b) find the values of c and d.

255. If $y=2x^3-3x-4$ find the value of $\dfrac{dy}{dx}$ when $x=2$

Hence find the equation of the tangent to the curve $y=2x^3-3x-4$ where $x=2$

256. Solve the simultaneous equations

$$x^2-2xy=15$$
$$3y+x=0$$

257. The area of a rectangular piece of carpet is 12 square metres and the perimeter is 19 metres. Find its length and breadth.

258. An oil tank is being filled such that it contains V litres of oil at time, t, seconds where $V = 10t^2 + 8t$
At what rate is the tank filling when t = 4 seconds. (Note: $\frac{dV}{dt}$ is the rate of change of volume.)

259. Find the equation of the perpendicular bisector of the line joining $(2, -5)$ and $(-1, -3)$

260. Differentiate $x^2 - 3x + 2$ from first principles.

261. Without working out the turning points sketch the curve $y = x^2(x-6)$

262. Sketch the curve $y = \frac{1}{x+3}$ State the equations of the asymptotes.

Answers

Page3

1. (a) $16 = 2^4 = 4^2$ and therefore $16^{1/4} = 2$ and $16^{1/2} = 4$
   (b) $27 = 3^3$ and $27^{1/3} = 3$
   (c) $64 = 8^2 = 4^3 = (2^2)^3 = 2^6$ and $8 = 64^{1/2}$ and $4 = 64^{1/3}$ and $2 = 64^{1/6}$
   (d) $81 = 9^2 = 3^4$ and $3 = 81^{1/4}$ and $9 = 81^{1/2}$

2. (a) $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$ (b) $4^{-1} = \frac{1}{4}$
   (c) $125^{1/3} = 5$ (d) $16^{1/4} = 2$
   (e) $16^{3/4} = (16^{1/4})^3 = (2)^3 = 8$
   (f) $32^{3/5} = (32^{1/5})^3 = 2^3 = 8$
   (g) $4^{5/2} = (4^{1/2})^5 = (2)^5 = 32$
   (h) $25^{3/2} = (25^{1/2})^3 = 5^3 = 125$
   (i) $\left(\frac{1}{3}\right)^{-2} = \frac{1}{\left(\frac{1}{3}\right)^2} = \frac{1}{1/9} = 9$
   (j) $\left(\frac{8}{27}\right)^{\frac{2}{3}} = \left(\frac{8^{\frac{1}{3}}}{27^{\frac{1}{3}}}\right)^2 = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$
   (k) $\frac{1}{3^{-2}} = 3^2 = 9$

Page4

3. Complete:…..$= 3^{\frac{1}{2}} \times 3^{-3} \times 3^5 = 3^{\frac{5}{2}}$

4. (a) $x\sqrt{x} = x^{\frac{3}{2}}$ or $\sqrt{x^3}$
   (b) $x^2\sqrt{x^3} = x^2x^{\frac{3}{2}} = x^{\frac{7}{2}}$ or $\sqrt{x^7}$
   (c) $\frac{x}{\sqrt{x}} = xx^{-\frac{1}{2}} = x^{\frac{1}{2}}$ or $\sqrt{x}$
   (d) $\frac{x^2}{\sqrt{x^3}} = x^2x^{-\frac{3}{2}} = x^{\frac{1}{2}}$ or $\sqrt{x}$
   (e) $\frac{\sqrt{x^5}}{x^2} = x^{\frac{5}{2}}x^{-2} = x^{\frac{1}{2}}$ or $\sqrt{x}$
   (f) $2x^3\sqrt{x^5} = 2x^3x^{\frac{5}{2}} = 2x^{\frac{11}{2}}$ or $2\sqrt{x^{11}}$
   (g) $\frac{x^2\sqrt{x}}{2x} = \frac{1}{2}x^2x^{\frac{1}{2}}x^{-1} = \frac{1}{2}x^{\frac{3}{2}}$ or $\frac{1}{2}\sqrt{x^3}$
   (h) $\frac{x\sqrt{x}}{x^{-2}} = x^1x^{\frac{1}{2}}x^2 = x^{\frac{7}{2}}$ or $\sqrt{x^7}$

5. $3(2 + 5\sqrt{2}) - 2(5 - 3\sqrt{2}) = 6+15\sqrt{2}-10+6\sqrt{2} = -4 + 21\sqrt{2}$

6. $(5 - \sqrt{3}) + 3(2 - \sqrt{2}) = 5 - \sqrt{3} + 6 - 3\sqrt{2} = 11 - \sqrt{3} - 3\sqrt{2}$

7. $(3 - 2\sqrt{5})(2 + \sqrt{5}) = 6-10-4\sqrt{5}+3\sqrt{5} = -4 - \sqrt{5}$

8. $(3 + \sqrt{2})^2 = (3+\sqrt{2})(3+\sqrt{2}) = 9+2+6\sqrt{2} = 11 + 6\sqrt{2}$

9. $(\sqrt{5} + 2\sqrt{5})(2\sqrt{3} - 3\sqrt{5}) = 6-30+4\sqrt{5}\sqrt{3}-3\sqrt{5}\sqrt{3} = -24 + \sqrt{15}$

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10. (a) $\sqrt{8} = \sqrt{4\times2} = \sqrt{4}\times\sqrt{2} = 2\sqrt{2}$
    (b) $\sqrt{18} = \sqrt{9\times2} = \sqrt{9}\times\sqrt{2} = 3\sqrt{2}$
    (c) $\sqrt{75} = \sqrt{3\times25} = \sqrt{3}\times\sqrt{25} = 5\sqrt{3}$
    (d) $\sqrt{360} = \sqrt{36\times10} = 6\sqrt{10}$
    (e) $\sqrt{32} = \sqrt{16\times2} = \sqrt{16}\times\sqrt{2} = 4\sqrt{2}$
    (f) $\sqrt{27} = \sqrt{9\times3} = \sqrt{9}\times\sqrt{3} = 3\sqrt{3}$
    (g) $\sqrt{98} = \sqrt{49\times2} = 7\sqrt{2}$
    (h) $15\sqrt{0.44} = 15\sqrt{\frac{44}{100}} = 15\frac{\sqrt{44}}{\sqrt{100}} = \frac{15\times2\sqrt{11}}{10} = 3\sqrt{11}$

11. $\frac{5}{\sqrt{2}} = \frac{5\sqrt{2}}{\sqrt{2}\sqrt{2}} = \frac{5\sqrt{2}}{2}$

12. $\frac{12}{\sqrt{3}} = \frac{12\sqrt{3}}{\sqrt{3}\sqrt{3}} = \frac{12\sqrt{3}}{3} = 4\sqrt{3}$

13. $\frac{2+3\sqrt{5}}{\sqrt{5}} = \frac{2\sqrt{5}+3\sqrt{5}\sqrt{5}}{\sqrt{5}\sqrt{5}} = \frac{2\sqrt{5}+15}{5}$ or $\frac{2\sqrt{5}}{5} + 3$

14. $\frac{2-\sqrt{2}}{\sqrt{2}} = \frac{2\sqrt{2}-\sqrt{2}\sqrt{2}}{\sqrt{2}\sqrt{2}} = \frac{2\sqrt{2}-2}{2} = \sqrt{2} - 1$

15. $\frac{12 - \sqrt{15} + 4\sqrt{5} - 3\sqrt{3}}{13}$

16. $= \frac{4\,(2-\sqrt{5})}{(\sqrt{5}+2)(2-\sqrt{5})} = \frac{8-4\sqrt{5}}{4-5} = 4\sqrt{5} - 8$

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17. $5(3x - 7) = 15x - 35$
18. $-(2x - 3) = -2x + 3$
19. $-2(6x + 11) = -12x - 22$
20. $(2x + 7)(3x - 5) = 6x^2 - 10x + 21x - 35 = 6x^2 + 11x - 35$
21. $(x - 4)(2x - 5) = 2x^2 - 8x - 5x + 20 = 2x^2 - 13x + 20$
22. $(2x - 3)^2 = (2x - 3)(2x - 3) = 4x^2 - 6x - 6x + 9 = 4x^2 - 12x + 9$
23. $(x^2 - 3x + 4)(2x^2 - 5x + 1) = 2x^4 - 5x^3 + x^2 - 6x^3 + 15x^2 - 3x + 8x^2 - 20x + 4 = 2x^4 - 11x^3 + 24x^2 - 23x + 4$
24. $(x + 2)(x - 5)(x - 1) = (x + 2)(x^2 - 6x + 5) = x^3 - 6x^2 + 5x + 2x^2 - 12x + 10 = x^3 - 4x^2 - 7x + 10$
25. $2x + 1 = Ax + A + Bx - 2B$
    Equating coefficients of x: $2 = A + B$
    Equating constant terms: $1 = A - 2B$
    Subtract $1 = 3B \Rightarrow B = 1/3$
    Substitute $2 = A + 1/3 \Rightarrow A = 5/3$
26. $2x^2 + 8x + 5 \equiv 2x^2 + 4ax + 2a^2 + b$
    Equating coefficients of x: $8 = 4a \quad \therefore a = 2$
    Equating constant terms: $5 = 2a^2 + b$
    Substitute for a: $5 = 8 + b \Rightarrow b = -3$

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27. (a) $4x(3y+1)$ (b) $4(2x - 3)$
    (c) $3ab(8b - 5a)$ (d) $2bc(3a + 4)$
    (e) $3pq^2(3p - 2q)$ (f) $z(z - 1)$
28. (a) $3(2x + 3)$ (b) $3(4x - 3)$
    (c) $x(x - 5)$ (d) $2x(x + 4)$
29. (a) $(x - 1)(x + 5)$
    (b) $(x + 2)[3(x - 1) - (x - 3)] = (x + 2)(3x - 3 - x + 3) = (x + 2)(2x) = 2x(x + 2)$
    (c) $2x(x + 1)[x + 4(x - 1)] = 2x(x + 1)(5x - 4)$
    (d) $(x + 2)(1 - x) + x(1 - x)(x + 3) = (1 - x)(x + 4 + x(x + 3)) = (1 - x)(x^2 + 4x + 4) = (1 - x)(x + 2)^2$

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30. $x^2 - 3x = x(x - 3)$
31. $2x^2 - 6x = 2x(x - 3)$
32. $x^2 - 7 = (x + \sqrt{7})(x - \sqrt{7})$
33. $x^2 - x - 20 = (x + 4)(x - 5)$
34. $x^2 + 7x + 12 = (x + 3)(x + 4)$
35. $x^2 - 10x + 16 = (x - 2)(x - 8)$
36. $3x^2 + 2x - 8 = (3x - 4)(x + 2)$
37. $6x^2 - 4x - 2 = (6x + 1)(x - 1)$
38. $8x^2 - 2x - 3 = (4x - 3)(2x + 1)$
39. $12x^2 + 4x - 5 = (6x + 5)(2x - 1)$
40. $9x^2 - 16 = (3x - 4)(3x + 4)$
41. $4x^2 - 121 = (2x - 11)(2x + 11)$

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42. $\frac{3x+9}{x^2-9} = \frac{3(x+3)}{(x+3)(x-3)} = \frac{3}{x-3}$
43. $\frac{2(x+1)}{(x-2)(x+1)} + \frac{(x-2)}{(x+1)(x-2)} = \frac{2x+2+x-2}{(x-2)(x+1)} = \frac{3x}{(x-2)(x+1)}$
44. $\frac{2x(x-5)}{6(x-5)} = \frac{2x}{6} = \frac{x}{3}$
45. $\frac{x-2}{2(x+2)(x-2)} = \frac{1}{2(x+2)}$
46. $\frac{(x+3)(x-2)}{2(2x^2-3x-2)} = \frac{(x+3)(x-2)}{2(2x+1)(x-2)} = \frac{x+3}{2(2x+1)}$
47. $\frac{3(x+4)}{(x+4)(x-4)} = \frac{3}{x-4}$
48. $\frac{x+2}{x+3} \times \frac{1}{3(x+2)} = \frac{1}{3(x+3)}$
49. $\frac{2(x-3)}{1} \times \frac{x+3}{2x} = \frac{(x-3)(x+3)}{x} = \frac{x^2-9}{x}$
50. $\frac{4(x-2)+3(x+1)}{12} = \frac{4x-8+3x+3}{12} = \frac{7x-5}{12}$
51. $\frac{2(x+1)}{(x+1)^2} + \frac{1}{(x+1)^2} = \frac{2x+2+1}{(x+1)^2} = \frac{2x+3}{(x+1)^2}$
52. $\frac{2\times4(x-1)}{12(x+4)(x-1)} - \frac{3\times3(x+4)}{12(x-1)(x+4)} = \frac{8x-8-9x-36}{12(x+4)(x-1)} = \frac{-x-44}{12(x+4)(x-1)}$
53. $\frac{2x^2}{x^2} + \frac{3}{x^2} = \frac{2x^2+3}{x^2}$
54. $\frac{x(2x+1)}{2x+1} - \frac{2}{2x+1} = \frac{2x^2+x-2}{2x+1}$
55. $\frac{x}{2x^2} + \frac{2}{2x^2} = \frac{x+2}{2x^2}$
56. $\frac{x^2}{x^2} + \frac{x}{x^2} + \frac{1}{x^2} = \frac{x^2+x+1}{x^2}$
57. $\frac{(1-x^2)+(1+x^2)(1+x^2)}{(1+x^2)(1-x^2)} = \frac{2+x^2+x^4}{1-x^4}$

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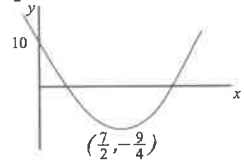
58. $x^2 - 4x + 3 = (x - 2)^2 - 4 + 3 = (x - 2)^2 - 1$
59. $3x^2 - 12x + 5 = 3[x^2 - 4x + 5/3] = 3[(x - 2)^2 - 4 + 5/3] = 3[(x - 2)^2 - 7/3] = 3(x - 2)^2 - 7$
60. $2x^2 + 10x - 5 = 2[x^2 + 5x - 5/2] = 2[(x + 5/2)^2 - 25/4 - 5/2] = 2[(x + 5/2)^2 - 35/4] = 2(x + 5/2)^2 - 35/2$
61. (a) $(x + 7)(x - 1)$, $(x + 3)^2 - 16$
    (b) $(2x + 3)(2x - 1)$, $4(x + 1/2)^2 - 4$
62. $y = (x - \frac{7}{2})^2 - \frac{49}{4} + 10$
    $y = (x - \frac{7}{2})^2 - \frac{9}{4}$



63. $-[x^2 + 6x - 10] = -[(x + 3)^2 - 9 - 10] = -[(x + 3)^2 - 19] = -(x + 3)^2 + 19$
    Turning point $(-3, 19)$

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64. $2x^2 - 5x = 0$
    $x(2x - 5) = 0$
    either $x = 0$ or $2x - 5 = 0$
    $\therefore x = 0$ or $5/2$
65. $x^2 = 7/4$
    $x = \frac{\sqrt{7}}{2}$ or $-\frac{\sqrt{7}}{2}$
66. $x^2 - 7x + 10 = 0$
    $(x - 5)(x - 2) = 0$
    either $x - 5 = 0$ or $x - 2 = 0$
    $x = 5$ or 2
67. $(2x + 3)(x - 2) = 0$
    either $2x + 3 = 0$ or $x - 2 = 0$
    $x = -3/2$ or 2
68. $x(x - 1) = 0$
    either $x = 0$ or $x - 1 = 0$
    $x = 0$ or 1
69. $x^2 = 2$
    $x = \pm\sqrt{2}$
70. $(x - 5)^2 - 25 + 5 = 0$
    $(x - 5)^2 = 20$
    $x - 5 = \pm\sqrt{20} = \pm 2\sqrt{5}$
    $x = 5 \pm 2\sqrt{5}$

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71. $x^2 + 6x - 7/2 = 0$
    $(x + 3)^2 - 9 - 7/2 = 0$
    $(x + 3)^2 = 25/2$
    $x + 3 = \pm\sqrt{\frac{25}{2}} = \pm\frac{5}{\sqrt{2}} = \pm\frac{5\sqrt{2}}{2}$
    $x = -3 \pm \frac{5\sqrt{2}}{2}$
72. $x^2 - 2x - 5 = 0$
    $a = 1,\ b = -2,\ c = -5$
    $x = \frac{--2 \pm \sqrt{(-2)^2 - 4.1.(-5)}}{2.1}$
    $x = \frac{2 \pm \sqrt{4 + 20}}{2} = \frac{2 \pm \sqrt{24}}{2}$
    $x = \frac{2 \pm 2\sqrt{6}}{2} = 1 + \sqrt{6}$ or $1 - \sqrt{6}$

73. $2x^2 + 3x - 4 = 0$

$x = \frac{-3\pm\sqrt{9+32}}{4} = \frac{-3\pm\sqrt{41}}{4}$

74. $2x^2 - 3x - 6 = 0$

$x = \frac{3\pm\sqrt{9+48}}{4} = \frac{3\pm\sqrt{57}}{4}$

$x = 2.64$ or -1.14 to 2 dec. pl.

75. $3x^2 - 12x + 7 = 0$

$x = \frac{12\pm\sqrt{144-84}}{6} = \frac{12\pm\sqrt{60}}{6}$

$x = 3.29$ or 0.71 to 2 dec. pl.

76. $x^2 + 2x = 15$

$x^2 + 2x - 15 = 0$

$(x+5)(x-3) = 0$

$x = -5$ or 3

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77. $(x+1)(x-1) = (2x+3)(2x-1)$

$x^2 - 1 = 4x^2 + 4x - 3$

$3x^2 + 4x - 2 = 0$

$x = \frac{-4\pm\sqrt{16-4\times3\times(-3)}}{2\times3} = \frac{-4\pm\sqrt{40}}{6}$

$= 0.387$ or -1.72 to 3 sig.fig.

78. $2(x-2) + 3(x+1) = 2(x+1)(x-2)$

$2x - 4 + 3x + 3 = 2x^2 - 2x - 4$

$0 = 2x^2 - 7x - 3$

$x = \frac{7\pm\sqrt{49-4\times2\times(-3)}}{2\times2} = \frac{7\pm\sqrt{73}}{4}$

$= 3.89$ or -0.386 to 3 sig.fig.

79. $1 = 4(x^2 + 4x + 4)$

$0 = 4x^2 + 16x + 16 - 1$

$= 4x^2 + 16x + 15$

$x = \frac{-16\pm\sqrt{16\times16-4\times4\times15}}{2\times4} = \frac{-16\pm\sqrt{16}}{8}$

$= -1.5$ or -2.5

80. $\frac{x}{x+3} = \frac{3}{x-2}$

$x^2 - 2x = 3x + 9$

$x^2 - 5x - 9 = 0$

$x = \frac{5\pm\sqrt{25-4\times1\times(-9)}}{2\times1} = \frac{5\pm\sqrt{61}}{2}$

$= -1.41$ or 6.41 to 2 dec. pl.

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81. $(x^2)^2 - 2(x^2) - 3 = 0$

$t^2 - 2t - 3 = 0$

$(t-3)(t+1) = 0$

$t = 3$ or -1

$x^2 = 3$ or $x^2 = -1$ (no real solutions)

$\therefore\ x = \pm\sqrt{3}$

82. $(y-2)(y-10) = 0$

$y = 2$ or 10

$\therefore\ x^2 + 1 = 2$ or $x^2 + 1 = 10$

$\therefore\ x^2 = 1$ or $x^2 = 9$

$x = \pm1$ or ±3

83. $\left(x^{\frac{1}{3}}\right)^2 - 7\left(x^{\frac{1}{3}}\right) - 8 = 0$

$t^2 - 7t - 8 = 0$

$(t-8)(t+1) = 0$

$t = -1$ or 8

$x^{\frac{1}{3}} = -1$ or 8 $\quad x = -1$ or 512

84. $2x - 7\sqrt{x} + 3 = 0$

$2\left(x^{\frac{1}{2}}\right)^2 - 7\left(x^{\frac{1}{2}}\right) + 3 = 0$

$2t^2 - 7t + 3 = 0$

$(2t-1)(t-3) = 0$

$t = ½$ or 3 so $x^{\frac{1}{2}} = ½$ or 3

$x = ¼$ or 9

85. Area of lawn $= x^2$ Area of path $= 1\times x = x$

Therefore $x = 1/10\ (x^2)$

$10x = x^2$

$0 = x^2 - 10x$

$x(x - 10) = 0$

$x = 0$ or 10 $\quad\therefore x = 10$

Area of lawn = 100 m^2 Area of path = 10 m^2

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86. $x^2 + (3x+3)^2 = (3x+4)^2$

$x^2 + 9x^2 + 18x + 9 = 9x^2 + 24x + 16$

$x^2 - 6x - 7 = 0$

$(x-7)(x+1) = 0$

$x = 7$ (or -1 not possible)

sides are 7, 24 and 25

87. (a) $b^2 - 4ac = 1 - 4\times2\times(-8) = 1 + 64 = 65$

$\therefore$ Two distinct roots.

(b) $b^2 - 4ac = 16 - 4\times1\times4 = 0$

$\therefore$ Two equal roots

(c) $b^2 - 4ac = 25 - 4\times3\times2 = 1$

$\therefore$ Two distinct roots.

(d) $b^2 - 4ac = 9 - 4\times1\times5 = 9 - 20 = -11$

$\therefore$ No real roots.

88. $k^2 - 4.2.8 = 0$

$k^2 = 64$

$k = 8$ or -8

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89. $y = x^2 + 6x + 5$

$y = (x+3)^2 - 9 + 5$

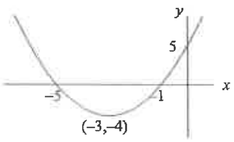
$y = (x+3)^2 - 4$

$x = 0 \quad y = 5$

$y = 0, \quad 0 = x^2 + 6x + 5$

$= (x+5)(x+1)$

$x = -5$ or -1



90. $y = x^2 + 4x + 3$

$y = (x+2)^2 - 4 + 3$

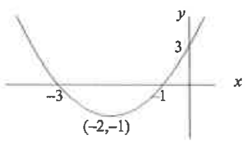
$y = (x+2)^2 - 1$

$x = 0, \quad y = 3$

$y = 0, \quad 0 = x^2 + 4x + 3$

$= (x+1)(x+3)$

$x = -1$ or -3



91. $y = x^2 - x - 2$

$y = (x - {}^1/_2)^2 - {}^1/_4 - 2$

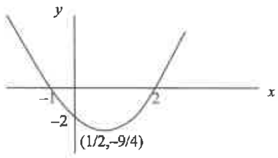
$y = (x - {}^1/_2)^2 - {}^9/_4$

$x = 0 \quad y = -2$

$y = 0, \quad 0 = x^2 - x - 2$

$= (x-2)(x+1)$

$x = 2$ or -1



92. $2s = 2ut + at^2$

$2s - 2ut = at^2$

$\frac{2s - 2ut}{t^2} = a$

$a = \frac{2s - 2ut}{t^2}$

93. $n(m+2) = 3-m$

$nm + 2n = 3-m$

$nm + m = 3-2n$

$m(n+1) = 3-2n$

$m = \frac{3-2n}{n+1}$

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94 $a\sqrt{x} - \sqrt{x} = c - b$

$\sqrt{x}(a-1) = c - b$

$\sqrt{x} = \frac{c-b}{a-1} \quad x = \frac{(c-b)^2}{(a-1)^2}$

95. $(x-2)^2 - 4 + 10 = (x-2)^2 + 6$

So $y = (x-2)^2 + 6$

$y - 6 = (x-2)^2$

$\pm\sqrt{y-6} = x - 2$

$x = 2 \pm \sqrt{y-6}$

96. $2x - 4y = -22$ …………(iii)

(i) – (iii) $7y = 28$

$y = 4$

$x - 8 = -11$

$x = -3$ Answer $x = -3, y = 4$

97. $6x + 2y = 14$

Add $8x = 24$

$x = 3$

Substitute $9 + y = 7$

$y = -2$

Answer $x = 3, y = -2$

98. $12x - 16y = 56$

$12x - 15y = 51$

Subtract $-y = 5$

$y = -5$

$3x + 20 = 14$

$x = -2$

Answer $x = -2, y = -5$

99. $y = 3x - 1$

Subst. $4x + 3(3x - 1) = 10$

$4x + 9x - 3 = 10$

$13x = 10 + 3$

Answer $x = 1, y = 2$

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100. $y = 7 - 3x$

Subst. $3x - 2(7 - 3x) = 13$

$3x - 14 + 6x = 13$

$9x = 27$

Answer $x = 3, y = -2$

101. $x = 5 + 3y$

Subst. $2(5 + 3y) - 5y = 8$

$10 + 6y - 5y = 8$

$y = -2$

$x = 5 - 6 = -1$

Answer $x = -1, y = -2$

102. $q = 1 - 2p$

Subst. $p(1-2p) + (1-2p)^2 = 3$

$p - 2p^2 + 4p^2 - 4p + 1 = 3$

$2p^2 - 3p - 2 = 0$

$(2p+1)(p-2) = 0$

$p = 2$ or $-1/2$

When $p = 2, q = -3$

When $p = -1/2, q = 2$

103. $x = 2y - 5$

Subst. $(2y-5)^2 + y^2 = 25$

$4y^2 - 20y + 25 + y^2 = 25$

$5y^2 - 20y = 0$

$5y(y-4) = 0$

$y = 0$ or 4

When $y = 0, x = -5$ When $y = 4, x = 3$

Ans: $x = -5, y = 0$ or $x = 3, y = 4$

104. $y = 2 - 2x$

Subst. $2 - 2x = x^2 - x - 2$

$0 = x^2 + 3x - 4$

$x = (x-1)(x+4)$

$x = 1$ or -4

When $x = 1, y = 2 - 2 = 0$

When $x = -4, y = 2 + 8 = 10$

Ans: $x = 1, y = 0$ or $x = -4, y = 10$

105. $2x - 5x < 5$

$-3x < 5$

$x > -5/3$

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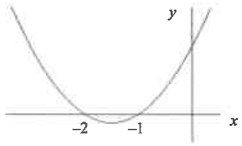
106. $y = x^2 + 3x + 2$

$0 = (x+2)(x+1)$

$x = -2$ or -1



$x \le -2$ or $x \ge -1$

107. Complete: $(k+1)^2 - 4.1.(k+4) \ge 0$

$k^2 + 2k + 1 - 4k - 16 \ge 0$

$k^2 - 2k - 15 \ge 0$

$(k-5)(k+3) \ge 0$

$k \le -3$ or $k \ge 5$

108. $p^2 - 2p - 3 \le 0$

$(p+1)(p-3) \le 0$

Smallest $p = -1$ largest $p = 3$

109. $-x^2 + 7x - 10 > 0$

$-(x^2 - 7x + 10) > 0$

$-(x-2)(x-5) > 0$

Answer: $2 < x < 5$

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110. (a) $\sqrt{8}=\sqrt{4\times 2}=2\sqrt{2}$

(b) $x=8^3=512$

(c) $8^{-\frac{2}{3}}=\frac{1}{8^{\frac{2}{3}}}=\frac{1}{\left(8^{\frac{1}{3}}\right)^2}=\frac{1}{2^2}=\frac{1}{4}$

(d) $8^x=\left(2^3\right)^x=2^{3x}$

111. (i) $\left(2+\sqrt{3}\right)\left(2-\sqrt{3}\right)=4-\sqrt{3}^{\,2}=4-3=1$

(ii) $\frac{3}{2+\sqrt{3}}=\frac{3\left(2-\sqrt{3}\right)}{\left(2+\sqrt{3}\right)\left(2-\sqrt{3}\right)}=\frac{6-3\sqrt{3}}{1}=6-3\sqrt{3}$

112. $\sqrt{63}=\sqrt{9\times 7}=3\sqrt{7}$

113. $\frac{x^{-1}}{\left(x^2\right)^{-1}}=\frac{x^{-1}}{x^{-2}}=x^{-1--2}=x$

114. (a) (i) $3^{\frac{1}{2}}$ (ii) 3^{3x}

(b) $3^{3x}\times 3^{x-1}=3^{\frac{1}{2}}$

$3^{3x+x-1}=3^{\frac{1}{2}}$

$\therefore\ 4x-1=½ \qquad x=\frac{3}{8}$

115. $\frac{5\left(\sqrt{7}-2\right)}{\left(\sqrt{7}+2\right)\left(\sqrt{7}-2\right)}=\frac{5\sqrt{7}-10}{7-4}=\frac{5}{3}\sqrt{7}-\frac{10}{3}$

116. $2^x=2^2 2^{\frac{1}{2}}=2^{\frac{5}{2}}$

$\therefore\ x=\frac{5}{2}$

117. $3^p=\left(3^3\right)^{q+1}=3^{3q+3}$

$p=3q+3$

118. $x-2\times 3x^{\frac{3}{2}}x^{\frac{1}{2}}+9x^3=x-6x^2+9x^3$

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119. $16xy=2^4\times 2^m\times 2^{2n}=2^{4+m+2n}$

120. $5x^2-6$ 121. $8x^3y+y^2-2x^2y$

122. $3x^4+15x^3-11x^2-10x+6$

123. x^2-6x+9 124. $5x\,(2x^2-x+4)$

125. $(y-3)(y+3)$ 126. $2x(x-4)(x+4)$

127. $(x+4)(x+2)$ 128. $(x-5)(x+3)$

129. $(5x-9)(x+2)$ 130. $(5x+6)(x-3)$

131. $(x-2)[x-2-5(x-1)]=(x-2)[x-2-5x+5]$

$=(x-2)\,(-4x+3)$

$=(x-2)\,(3-4x)$

132. (a) $3x$ (b) xy (c) $\frac{3a}{2b}$

133. (a) $\frac{1}{x+1}$ (b) $\frac{x(x+3)}{(x+3)(x-5)}=\frac{x}{x-5}$

(c) $\frac{(x+1)(x-1)}{(x-1)(x-6)}=\frac{x+1}{x-6}$

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134. (a) $\frac{8}{x}$ (b) $\frac{xy-2}{2y}$

(c) $\frac{3y}{xy}+\frac{2x}{xy}=\frac{3y+2x}{xy}$

(d) $\frac{4x}{2x(x-1)}+\frac{3(x-1)}{2x(x-1)}=\frac{4x+3x-3}{2x(x-1)}=\frac{7x-3}{2x(x-1)}$

(e) $\frac{x^2}{3x}+\frac{6}{3x}=\frac{x^2+6}{3x}$

(f) $\frac{(x-1)}{(x+1)(x-1)}-\frac{2(x+1)}{(x+1)(x-1)}=\frac{-x-3}{x^2-1}=\frac{x+3}{1-x^2}$

135. $(x+3)^2-1$

$(x+3)^2=1$

$x+3=\pm 1$

$x=-3\pm 1$

$x=-4$ or -2

136. $3(x-4)^2-24$

137. $x(x-5)=0 \Rightarrow x=0$ or 5

138. $x^2=5 \Rightarrow x=+\sqrt{5}$ or $-\sqrt{5}$

139. $(x-4)(x+3)=0 \quad\Rightarrow\quad x=4$ or -3

140. $x^2+2x-7=0$

$(x+1)^2-8=0$

$x=-1\pm\sqrt{8}=-1\pm 2\sqrt{2}$

141. $x=\frac{2\pm\sqrt{4+4}}{2}=\frac{2\pm\sqrt{8}}{2}=\frac{2\pm 2\sqrt{2}}{2}$

$=1+\sqrt{2}$ or $1-\sqrt{2}$

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142. $4x^2-20x+25=2x$

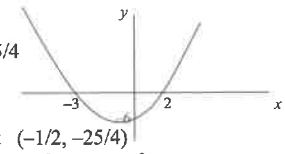
$4x^2-22x+25=0$

$x=\frac{22\pm\sqrt{484-400}}{8}=\frac{22\pm\sqrt{84}}{8}=\frac{22\pm 2\sqrt{21}}{8}=\frac{11\pm\sqrt{21}}{4}$

143.

$y=(x+3)(x-2)$

$y=(x+½)^2-25/4$



min at $(-1/2,-25/4)$

144. $a(x+b)^2+c=a(x^2+2bx+b^2)+c$

$=ax^2+2abx+ab^2+c$

eq coeff of x^2: $a=2$

eq coeff of x: $2ab=12 \Rightarrow b=3$

eq cont. terms: $16=ab^2+c \Rightarrow 16=18+c \Rightarrow c=-2$

$2(x+3)^2-2$

145. $-2x^2+3x\le 0$

$x(-2x+3)\le 0$

$x\le 0$ or $x\ge 3/2$

146. $x^2+3x-10=0$

$(x+5)(x-2)=0$

$x=2$ or -5

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147. $x^2-6x+p=x^2-2qx+q^2+4$

$q=3,\ p=9+4=13$

148. $y=4+x$

$(4+x)^2-5x^2=20$

$16+8x+x^2-5x^2=20$

$4x^2-8x+4=0$

$x^2-2x+1=0$

$(x-1)(x-1)=0$

$x=1$ twice

$x=1,\ \ y=5$

Only one solution and so the line touches curve.

149. $t^2-5t+6=0$

$(t-3)(t-2)=0$

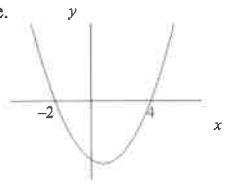
$t=2$ or 3

$x^{\frac{1}{2}}=2$ or 3

$x=4$ or 9

150. $x^2-2x-8>0$

$(x-4)(x+2)$



$x<-2$ or $x>4$

151. $y(x-1)=x+1$

$yx-y=x+1$

$yx-x=y+1$

$x(y-1)=y+1$

$x=\frac{y+1}{y-1}$

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152. Gradient of line $=(1-5)/(3--1)=-1$

Gradient of a perpendicular line = 1

Midpoint of line = (1,3)

Length of line = $\sqrt{(3--1)^2+(1-5)^2}=\sqrt{32}=4\sqrt{2}$

153. $\sqrt{(5-1)^2+(-10--1)^2}=\sqrt{16+81}=\sqrt{97}$

$=9.8$ to 1 dec. pl.

154. $2y=3-5x$ so $y=-(5/2)x+(3/2)$

155. $3y=6x-4$ so $y=2x-(4/3)$

156. $3y=2x+15$

$2x-3y+15=0$

157. $6y=10-9x$

$9x+6y-10=0$

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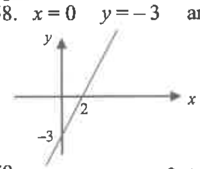
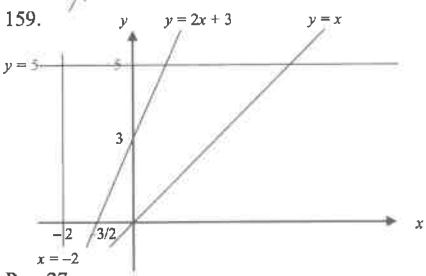
158. $x=0 \quad y=-3$ and $y=0 \quad x=2$



159.



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160. $y-5=3\,(x-2)$

$y-5=3x-6 \quad\Rightarrow\quad y=3x-1$

161. $m=\frac{5-(-3)}{1-(-2)}=\frac{8}{3}$

$y-5=\frac{8}{3}\,(x-1)$

$3y-15=8x-8 \quad\Rightarrow\quad 3y=8x+7$

162. $y=x/2+5/2$ Gradient ½

$(y--3)=½\,(x-2)$

$2y+6=x-2$

$2y-x+8=0$

163. Gradient 2 Perpendicular gradient –½

Equation of line $(y--3)=-½\,(x-5)$

$2y+6=-x+5$

$2y+x+1=0$

164. Gradient $\frac{7-(-2)}{-1-8}=\frac{9}{-9}=-1$

Perpendicular gradient 1

Equation of line $(y--2)=1\,(x--1)$

$y+2=-x+1$

$y=x-1$

Page28

165. $4(3x-4)+2x=7$

$12x-16+2x=7$

$14x=23$

$x=23/14 \quad y=(69/14)-4=13/14$

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166. $d\propto t$

$d=kt$

$15=k\times 4$

$k=\frac{15}{4}$

$d=\frac{15}{4}\,t$

(i) $d=\frac{15}{4}\times 12=45$ m

(ii) $60=\frac{15}{4}t$

$t=16$ seconds

167. $d\propto t^2$

$d=kt^2$

$705.6=k\times 144$

$k=4.9$

$s=4.9\,t^2$

$122.5=4.9\,t^2$

$t=5$ seconds

168. $T\propto\sqrt{l}$

$T=k\sqrt{l}$

$2.45=k\times\sqrt{1.5}$

$k=\frac{2.45}{\sqrt{1.5}}$

$T=\frac{2.45}{\sqrt{1.5}}\sqrt{l}$

If $l=2$ m, $T=\frac{2.45}{\sqrt{1.5}}\sqrt{2}=2.83$ seconds

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169. $p\propto\frac{1}{q}$

$p=k\,\frac{1}{q}$

$2.5=k\times ½$

$k=5$

$p=5\times\frac{1}{q}$

If $q=25$, $p=5\times\frac{1}{25}=\frac{1}{5}$

170. $V\propto\frac{1}{p}$

$p=k\,\frac{1}{p}$

$5\times 10^{-5}=k\times\frac{1}{1.2\times 10^5}$

$k=6$

$V=6\times\frac{1}{p}$

$V=6\times\frac{1}{2.5\times 10^{-5}}=2.4\times 10^5$ N/m²

171. $a\propto\frac{1}{d^2}$

$a=k\,\frac{1}{d^2}$

$9.8\times 6371^2=k$

$a=9.8\times 6371^2\times\frac{1}{d^2}$

$a=9.8\times 6371^2\times\frac{1}{7371^2}$

$a=7.32\ \mathrm{ms^{-2}}$ to 3 sig. fig.

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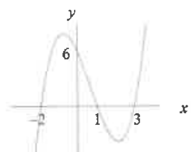
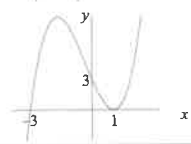
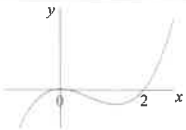
172. (a)



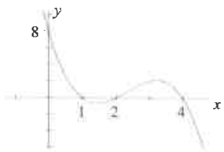
(b)



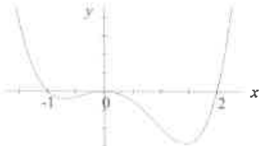
173. (a)



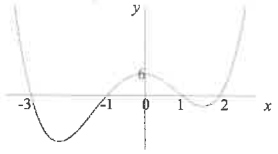
(b)



174. (a)



(b)



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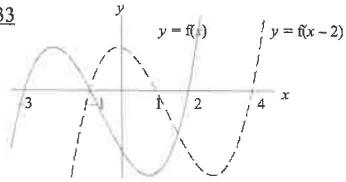
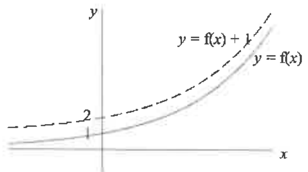
175.



$f(x-2) = (x-4)(x-1)(x+1)$

176.



$f(x) + 1 = 2^x + 1$

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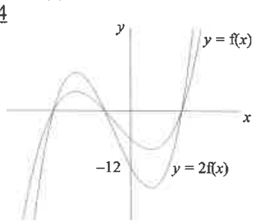
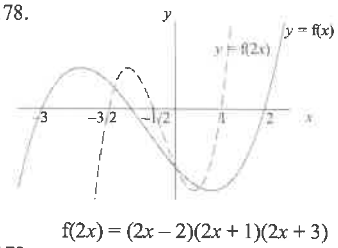
177.



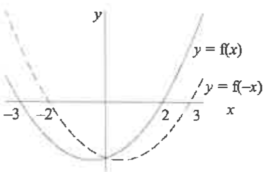
$2\ f(x) = 2(x-2)(x+1)(x+3)$

178.



$f(2x) = (2x-2)(2x+1)(2x+3)$

179.



$f(-x) = (-x)^2 + (-x) - 6$
$= x^2 - x - 6$

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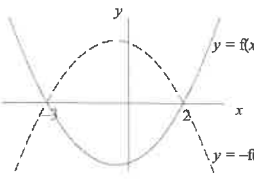
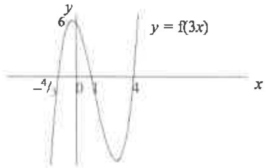
180.



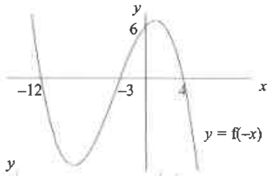
$-f(x) = -(x^2 + x - 6) = -x^2 - x + 6 = (2-x)(x+3)$

181.

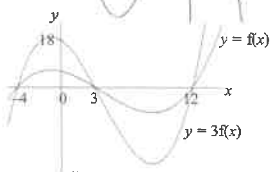
(a)



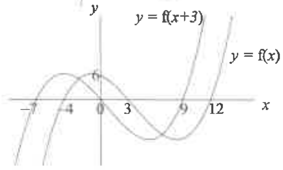
(b)



(c)



(d)



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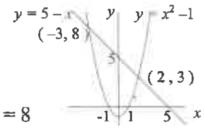
182. (a) Translation by vector $\begin{pmatrix}-1\\0\end{pmatrix}$

(b) Stretch by factor 2 parallel to y axis away from the x axis

183. Replace x with $(x+2)$

$(x+2)^2 - 3(x+2) + 5 = x^2 + 4x + 4 - 3x - 6 + 5$
$= x^2 + x + 3$

184. $5 - x = x^2 - 1$
$0 = x^2 + x - 6$
$= (x+3)(x-2)$
$x = -3$ or 2
$x = 2,\ y = 3$ or $x = -3,\ y = 8$



Points of intersection are $(-3, 8)$ and $(2, 3)$

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185. $\dfrac{y-11}{x--2} = \dfrac{7-11}{4--2} = \dfrac{-4}{6}$
$3y + 2x = 29$

186. Midpoint (5,5) Gradient – (4/8) = – ½ , grad of perpendicular 2

$\dfrac{y-5}{x-5} = 2 \quad\Rightarrow\quad y = 2x - 5$

187. $R \propto \frac{1}{d^2} \quad\Leftrightarrow\quad R = k\frac{1}{d^2}$

$0.02 \times 0.1^2 = k \quad\Rightarrow\quad k = 0.0002$

$R = 0.0002 \times \frac{1}{d^2}$

$R = 0.0002 \times \frac{1}{0.3^2} \quad\therefore\ R = 0.002$ to 1 sig. fig.

188. (a) $5y = 7x + 17$ (b) $7y = -5x + 39$

189. $2x - 7 = x^2 - 6$
$0 = x^2 - 2x + 1$
$b^2 - 4ac = 4 - 4 \times 1 \times 1 = 0$ therefore the line touches the curve.

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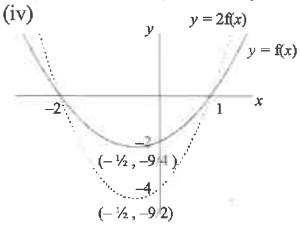
190. (i) $x = 0\ y = -2$,
$y = 0\ \ x^2 + x - 2 = 0 \quad (x+2)(x-1) = 0$
$x = 1$ or -2

(ii) $(x + ½)^2 - 9/4$ min $(-½, -9/4)$

(iii) & (iv)



191. $AB = \sqrt{(5-3)^2 + (3--1)^2} = \sqrt{20} = 2\sqrt{5}$

Mid point (4, 1)

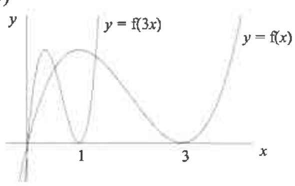
Gradient of AC $= \frac{2--1}{2-3} = -3$

Gradient of BC $= \frac{2-3}{2-5} = \frac{1}{3}$

$-3 \times ⅓ = -1$ therefore ACB is a right angle.

192. (i) $x(x-3)^2$

(ii)&(iii)



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193. $\dfrac{dy}{dx} = \lim\limits_{h\to 0} \dfrac{(x+h)^3 - x^3}{h}$

$= \lim\limits_{h\to 0} \dfrac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$

$= \lim\limits_{h\to 0} \dfrac{3x^2h + 3xh^2 + h^3}{h}$

$= \lim\limits_{h\to 0} \dfrac{3x^2h}{h} + \dfrac{3xh^2}{h} + \dfrac{h^3}{h}$

$= \lim\limits_{h\to 0} 3x^2 + 3xh + h^2$

$= 3x^2$

193. $\dfrac{dy}{dx} = \lim\limits_{h\to 0} \dfrac{\frac{1}{x+h} - \frac{1}{x}}{h}$

$= \lim\limits_{h\to 0} \dfrac{\frac{x-x-h}{x(x+h)}}{h}$

$= \lim\limits_{h\to 0} \dfrac{-h}{x^2h + xh^2}$

$= \lim\limits_{h\to 0} \dfrac{-1}{x^2 + xh}$

$= -\dfrac{1}{x^2}$

195. (a) $3x^2$ (b) $24x^3$ (c) $10x$ (d) $6x + 2$

196. (a) $dy/dx = 30x^4$
(b) $dy/dx = 4x^3 - 6x + 2$
when $x = 2$ $dy/dx = 32 - 12 + 2 = 22$

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197. $dy/dx = 2x - 4$
$8 = 2x - 4$
$x = 6 \quad\therefore$ Point is $(6, 12)$

198. $\frac{dv}{dt} = 6t - 2$

$t = 2,\ \frac{dv}{dt} = 6 \times 2 - 2 = 10ms^{-2}$

199. For side length x $V = x^3$, $\frac{dv}{dx} = 3x^2$ m³ per m

200. $y = \sqrt{x^3} = x^{\frac{3}{2}} \quad \frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x}$

201. $y = \frac{1}{x^2} = x^{-2} \quad \frac{dy}{dx} = -2x^{-3} = -\frac{2}{x^3}$

202. $f(x) = \frac{2}{\sqrt{x}} = 2x^{-1/2}$

$f'(x) = -\frac{1}{2} \times 2x^{-\frac{3}{2}} = -\frac{1}{\sqrt{x^3}}$

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203. $y = \frac{1}{x} - \sqrt{x} + 3x^2 = x^{-1} - x^{½} + 3x^2$

$\frac{dy}{dx} = -x^{-2} - \frac{1}{2}x^{-\frac{1}{2}} + 6x = -\frac{1}{x^2} - \frac{1}{2\sqrt{x}} + 6x$

204. $y = \sqrt[3]{x^2} = x^{\frac{2}{3}}$

$\frac{dy}{dx} = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{x}}$

205. $y = \frac{3}{2x} = \frac{3}{2}x^{-1}$

$\frac{dy}{dx} = \frac{3}{2}(-1)x^{-2} = -\frac{3}{2x^2}$

206. $y = \frac{2x^3 - 3x^2}{x} = \frac{2x^3}{x} - \frac{3x^2}{x} = 2x^2 - 3x$

$\frac{dy}{dx} = 4x - 3$

207 $y = \sqrt{x}(x+2) = x^{\frac{1}{2}}(x+2) = x^{\frac{3}{2}} + 2x^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} + \frac{2}{2}x^{-\frac{1}{2}} = \frac{3}{2}\sqrt{x} + \frac{1}{\sqrt{x}}$

208. $f'(x) = \frac{3}{2}x^{\frac{1}{2}} + 2$

209. $\frac{dP}{dt} = 100 \times \frac{1}{2} \times t^{-\frac{1}{2}} = \frac{50}{\sqrt{t}}$

when $t = 4$, $\frac{dP}{dt} = \frac{50}{\sqrt{4}} = 25$ birds per year

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210. $dy/dx = 6x^2 - 10x + 2$ When $x = 1$ $dy/dx = -2$
Gradient of tangent $= -2$
$y - 4 = -2(x - 1) = -2x + 2$
Equation of tangent is $y = -2x + 6$ or $2x + y - 6 = 0$
Gradient of normal = ½
$y - 4 = \frac{1}{2}(x - 1)$
$2y - 8 = x - 1$
Equation of normal is $x - 2y + 7 = 0$

211. $x = -1$, $y = 2 + 2 + 5 = 9$
$dy/dx = 4x - 2$
When $x = -1$, $dy/dx = -4 - 2 = -6$
Gradient of tangent $= -6$
$y - 9 = -6(x - -1) = -6x - 6$
$y = -6x + 3$ or $6x + y - 3 = 0$

212. $dy/dx = 2x - 1$
$8 = 2x - 1$
$x = 9/2$ Then $y = 81/4 - 18/4 - 24/4 = 39/4$
Gradient of normal $= -1/8$
$y - \frac{39}{4} = -\frac{1}{8}(x - \frac{9}{2})$
$y - \frac{39}{4} = -\frac{1}{8}x + \frac{9}{16}$
$y = -\frac{1}{8}x + \frac{165}{16}$ or $2x + 16y - 165 = 0$

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213. $x^5 + c$
214. $x^3 - x^2 + 4x + c$
215. $\int 4x^2 - 8x\,dx = \frac{4x^3}{3} - \frac{8x^2}{2} + c = \frac{4x^3}{3} - 4x^2 + c$
216. $5x - \frac{x^4}{4} + c$
217. $\int 2x^2 + x - 6\,dx = \frac{2}{3}x^3 + \frac{1}{2}x^2 - 6x + c$
218. $\int x^2 - 4\,dx = \frac{x^3}{3} - 4x + c$
219. $\int x^{\frac{1}{2}} + 3x^{-2} - 3x\,dx = \frac{2}{3}x^{\frac{3}{2}} - \frac{3}{x} - \frac{3}{2}x^2 + c$
220. $y = \int 3x^3 + x\,dx = \frac{3x^4}{4} + \frac{x^2}{2} + c$
221. $y = \frac{2}{5}x^{\frac{5}{2}} - \frac{1}{x^2} - 3x + c$

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222. $dy/dx = 3x^2 - 6x$
When $x = 2$ Gradient $= 3\times 4 - 12 = 0$

223. $dy/dx = 2x - 1$
(a) Curve cuts y axis at $x = 0$. Gradient $= -1$
(b) $y = (x - 4)(x + 3)$ cuts x axis at $x = -3$ and $x = 4$
Gradients are -7 and 7.

224. $dy/dx = 3x^2 - 12$.
If $3x^2 - 12 = 15$, $3x^2 = 27$, so $x^2 = 9$
$x = 3$ and $x = -3$. Substitute into equation
Coordinates (3, 0) and (– 3 , 18)

225. $y = 2x + 3\sqrt{x} - \frac{4}{x} = 2x + 3x^{\frac{1}{2}} - 4x^{-1}$
$\frac{dy}{dx} = 2 + \frac{3}{2}x^{-\frac{1}{2}} + 4x^{-2} = 2 + \frac{3}{2\sqrt{x}} + \frac{4}{x^2}$

226. $dy/dx = 4x + 5$
(a) $x = 2$ Gradient = 13
(b) At $x = 2$, $y = 14$ $(y - 14) = 13(x - 2)$
$y = 13x - 12$ is equation of tangent
(c) Gradient curve at $x = 1$ is 9.
Gradient normal is $-1/9$
(d) At $x = 1$, $y = 3$ $(y - 3) = -(1/9)(x - 1)$
$9y - 27 = -x + 1$
$x + 9y - 28 = 0$ is the equation of the normal

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227. $y = (x - 4)(x - 1)$ $y = 0$, $x = 1$ or 4
$dy/dx = 2x - 5$
at $x = 1$ Grad $= -3$, at $x = 4$ Grad $= 3$
at $x = 1$, $y = 0$: $y = -3(x - 1)$
Equation of tangent is $3x + y - 3 = 0$ (i)
at $x = 4$, $y = 0$: $y = 3(x - 4)$
Equation of tangent is $3x - y - 12 = 0$ (ii)
Intersection: (i) – (ii) $2y = -9$, $y = -9/2$
Substitute into (ii) $3x = 15/2$, $x = 5/2$
Point of intersection is (5/2 . – 9/2)

228. $y = x^3 - 13x + 12$
$dy/dx = 3x^2 - 13$
(a) at $x = 2$ Grad is $12 - 13 = -1$
$(y + 6) = -1(x - 2)$
$x + y + 4 = 0$ is equation of tangent
(b) $y = 0$, $x = -4$ $(-4, 0)$
(c) at $x = 1$, Grad is -10, Grad of normal is $1/10$
at $x = 1$, $y = 0$ Normal $y = (1/10)(x - 1)$
$x - 10y - 1 = 0$
(d) $x = -39/11$ and $y = -5/11$

229. (a) $\frac{dy}{dx} = 3x^2 - 5$
(b) $y = x^3 - x^2 - 20x$ $\frac{dy}{dx} = 3x^2 - 2x - 20$
(c) $y = 2x^{\frac{3}{2}}$ $\frac{dy}{dx} = 3x^{\frac{1}{2}} = 3\sqrt{x}$
(d) $y = 3x^{-2} + \frac{7}{2}x^{-\frac{1}{2}}$
$\frac{dy}{dx} = -6x^{-3} - \frac{7}{4}x^{-\frac{3}{2}} = -\frac{6}{x^3} - \frac{7}{4\sqrt{x^3}}$
(e) $y = \frac{2x^2}{x^{\frac{1}{2}}} - \frac{5x}{x^{\frac{1}{2}}} = 2x^{\frac{3}{2}} - 5x^{\frac{1}{2}}$
$\frac{dy}{dx} = 3x^{\frac{1}{2}} - \frac{5}{2}x^{-\frac{1}{2}} = 3\sqrt{x} - \frac{5}{2\sqrt{x}}$

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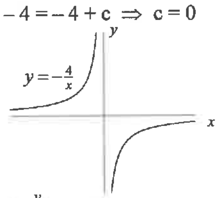
230. $\frac{3}{4}x^4 - 2x^2 - 5x + c$
231. $(1/3)x^3 + (3/2)x^2 - 4x + c$
232.
$$\int 4x^{\frac{1}{2}} - 2x^{-2}dx = 4\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{2x^{-1}}{-1} + c$$
$$= \frac{8}{3}x^{\frac{3}{2}} + 2x^{-1} + c$$
$$= \frac{8}{3}\sqrt{x^3} + \frac{2}{x} + c$$
233. $\int x^3 - 6x^2\,dx = (1/4)x^4 - 2x^3 + c$
234. $\int x^2 + 3\,dx = (1/3)x^3 + 3x + c$
235. $y = \int 3x^2 + 4x\,dx = x^3 + 2x^2 + c$
$x = 1$ $y = 7$ ∴ $7 = 1 + 2 + c \Rightarrow c = 4$
so $y = x^3 + 2x^2 + 4$
236. $\int 1 - t^{\frac{1}{3}}\,dt = t - \frac{3}{4}t^{4/3} + c$
237. $\int(3x + 1)^2\,dx = \int 9x^2 + 6x + 1\,dx$
$= 3x^3 + 3x^2 + x + c$
238. $y = \int 8x + 1\,dx = 4x^2 + x + c$
$x = 2$ $y = 5$ ∴ $5 = 16 + 2 + c \Rightarrow c = -13$
so $y = 4x^2 + x - 13$
239. $y = \int \frac{4}{x^2}dx = \int 4x^{-2}dx = -4x^{-1} + c$
$= -\frac{4}{x} + c$
$x = 1$, $y = -4$ ∴ $-4 = -4 + c \Rightarrow c = 0$
so $y = -\frac{4}{x}$



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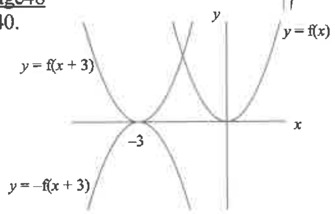
240.



$f(x + 3) = (x + 3)^2 = x^2 + 6x + 9$
$-f(x + 3) = -(x^2 + 6x + 9) = -x^2 - 6x - 9$

241. Grad $= -4$
$y - 5 = -4(x + 2) \Rightarrow 4x + y + 3 = 0$
242. $\frac{(2-2\sqrt{2}+1)(\sqrt{2}-1)}{(\sqrt{2}+1)(\sqrt{2}-1)} = \frac{-7+5\sqrt{2}}{2-1} = -7 + 5\sqrt{2}$
243. $2^{3x} = 2^{2x-2}$ ∴ $3x = 2x - 2$ $x = -2$
244. $\int \frac{1}{3}x^{\frac{3}{2}} - 2x^{-\frac{1}{2}}dx = \frac{2}{15}x^{\frac{5}{2}} - 4x^{\frac{1}{2}} + c$
245. $V \propto h^2 \Leftrightarrow V = k\,h^2$
$279 = k \times 64 \Rightarrow k = \frac{279}{64}$
∴ $V = \frac{279}{64}h^2$
$558 = \frac{279}{64}h^2$
∴ $h = \sqrt{128}$ ∴ $h = 11.3$ cm to 3 sig. fig.

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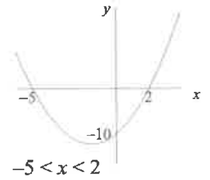
246. '$b^2 - 4ac$' $= k^2 - 4 \times 1 \times 4 = k^2 - 16 > 0$
$k < -4$ or $k > 4$
247. $x^2 - 6 = x + 6 \Rightarrow x^2 - x - 12 = 0$
$(x - 4)(x + 3) = 0 \Rightarrow x = 4$ or -3
Points of intersection are (4 , 10) and (– 3 , 3)
248.



Solution is $-5 < x < 2$

249. $y - 2x + 5 = 0 \Rightarrow y = 2x - 5$ ∴ grad = 2
perp grad = – ½
$y + 3 = -\frac{1}{2}(x - 1) \Rightarrow x + 2y + 5 = 0$
250. $\frac{1}{\sqrt[3]{8}} + \left(27^{\frac{1}{3}}\right)^2 = \frac{1}{2} + 3^2 = 9\frac{1}{2}$
251. (i) $y = x^{\frac{1}{2}} + x^{-\frac{1}{2}}$ $\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}}$
(ii) $y = x^4 - 2x^2$ $\frac{dy}{dx} = 4x^3 - 4x$
(iii) $y = 3x^2 + 2x$ $\frac{dy}{dx} = 6x + 2$
252. $b^2 - 4ac = 9 - 4 \times 2 \times 4 = 9 - 32 = -23$
negative and so no real roots.
253. $(x - 3)^2 - 4$ vertex $(3, -4)$

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254. (a) $6 - 2\sqrt{5} - 3\sqrt{5} + 5 = 11 - 5\sqrt{5}$
$a = 11$ $b = -5$
(b) $\frac{(2-\sqrt{5})(3-\sqrt{5})}{(3+\sqrt{5})(3-\sqrt{5})} = \frac{11-5\sqrt{5}}{9-5} = \frac{11}{4} - \frac{5}{4}\sqrt{5}$
$c = \frac{11}{4}$, $d = -\frac{5}{4}$
255. $dy/dx = 6x^2 - 3$ at $x = 2$ $dy/dx = 21$
$x = 2$, $y = 6$ $y - 6 = 21(x - 2)$
$y = 21x - 36$
256. $x = -3y$
$(-3y)^2 - 2(-3y)y = 15$
$9y^2 + 6y^2 = 15$
$y^2 = 1 \Rightarrow y = \pm 1$
Solutions $y = +1$ $x = -3$ or $y = -1$, $x = 3$
257. $xy = 12$ and $2x + 2y = 19$
∴ $y = 12/x$
substitute $2x + 24/x = 19$
$2x^2 + 24 = 19x$
$2x^2 - 19x + 24 = 0$
$(x - 8)(2x - 3) = 0$
$x = 8$ or 1.5
$x = 8$ $y = 1.5$ or $x = 1.5$ $y = 8$
i.e. length 8 breadth 1.5 or vice versa.

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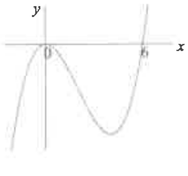
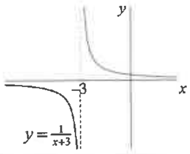
258. $\frac{dV}{dt} = 20t + 8$
at t = 4 rate is 88 litres per second.
259. Midpoint (0.5, – 4) grad – 2/3
grad of perp 3/2
$y + 4$_ = 3/2 $(x - 0.5)$ $4y = 6x - 19$
260. $\frac{dy}{dx} = \lim_{h\to 0} \frac{(x+h)^2 - 3(x+h) + 2 - (x^2 - 3x + 2)}{h}$
$= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - 3x - 3h + 2 - x^2 + 3x - 2}{h}$
$= \lim_{h\to 0} \frac{2xh + h^2 - 3h}{h}$
$= \lim_{h\to 0} 2x + h - 3$
$= 2x - 3$
261. (a) $y = x^2(x - 6)$
(b)



262.



Asymptotes are $x = -3$
and $y = 0$