Redhill A-Level Maths

Workbook AS Pure 2

ALGEBRA and FUNCTIONS

Factorising polynomials:

Polynomials are of the form $ax^n + bx^{n-1} + cx^{n-2} + \ldots\ldots\ldots\ldots$ for a positive integer n.
You should know how to factorise quadratics.

Factorising cubics with no constant term.

Example: Factorise $x^3 + 4x^2 - 5x$ $x^3 + 4x^2 - 5x = x(x^2 + 4x - 5)$ $= x(x+5)(x-1)$	*Take x out as a factor. Factorise the remaining quadratic.*
1. (i) Factorise $x^3 - 4x$ (ii) Without finding the turning points sketch $y = x^3 - 4x$	*Take x out as a factor.* *Factorise the remaining quadratic.* *Sketch the curve.*
2. $\mathrm{f}(x) = x^3 + 4x^2$ Factorise $\mathrm{f}(x)$ and hence sketch $y = \mathrm{f}(x)$ without finding the turning points.	*Take x^2 out as a factor.* *Sketch the curve.*
3. $\mathrm{g}(x) = x^3 - 2x^2 - 8x$ (i) Factorise $\mathrm{g}(x)$ (ii) Without finding the turning points sketch $y = \mathrm{g}(x)$	*Take x out as a factor.* *Factorise the remaining quadratic.* *Sketch the curve.*
4. Factorise $x^3 + 2x^2 - 5x$	*Take x out as a factor. The resulting quadratic does not factorise with integer constants.*
5. Factorise $4x^3 - 7x^2 - 2x$ and hence solve the equation $4x^3 - 7x^2 - 2x = 0$	*Take x out as a factor.* *Factorise the remaining quadratic.*
6. (i) Factorise $-x^3 - 2x^2 + 3x$ (ii) Without finding the turning points sketch $y = -x^3 - 2x^2 + 3x$	*Take $-x$ out as a factor. Careful with minuses. Check your answer by multiplying the $-x$ by the other terms (in your head).* *Factorise the remaining quadratic.* *Sketch the curve.*

Algebraic division:

Example: Divide $x^3 + 7x^2 - 36$ by $(x+3)$

x into x^3 or what times x equals x^3 ?
Answer: x^2

$$\begin{array}{r|l} & x^2 \\ \hline x+3 & x^3 + 7x^2 \quad - 36 \\ & -(x^3 + 3x^2) \\ \hline & \quad 4x^2 \end{array}$$

x^2 times $x+3$ and subtract

x into $4x^2$ or what times x equals $4x^2$?
Answer: $4x$

$$\begin{array}{r|l} & x^2 + 4x \\ \hline x+3 & x^3 + 7x^2 \quad - 36 \\ & -(x^3 + 3x^2) \\ \hline & \quad 4x^2 \quad - 36 \\ & \quad -(4x^2 + 12x) \\ \hline & \quad\quad -12x - 36 \end{array}$$

$4x$ times $x+3$ and subtract

x into $-12x$ or what *times x equals* $-12x$?
Answer: -12

$$\begin{array}{r|l} & x^2 + 4x - 12 \\ \hline x+3 & x^3 + 7x^2 \quad - 36 \\ & -(x^3 + 3x^2) \\ \hline & \quad 4x^2 \quad - 36 \\ & \quad -(4x^2 + 12x) \\ \hline & \quad\quad -12x - 36 \\ & \quad\quad -(-12x - 36) \\ \hline & \quad\quad ---- \end{array}$$

-12 times $x+3$ and subtract

Remainder

Note:
Quotient is $x^2 + 4x - 12$
Remainder is 0

$\therefore\ x^3 + 7x^2 - 36 = (x+3)(x^2 + 4x - 12) = (x+3)(x+6)(x-2)$

7. $2x^3 - 5x^2 - 23x - 10$ divided by $(x-5)$
COMPLETE:

Quotient

$$\begin{array}{r|l} & \\ \hline x - 5 & 2x^3 - 5x^2 - 23x - 10 \\ & -(\qquad\qquad) \\ & \qquad -(\qquad\qquad) \\ & \qquad\qquad -(\qquad\qquad) \end{array}$$

Remainder

8. Find the quotient and remainder when $x^3 - 3x^2 - 22x + 12$ is divided by $(x-1)$

Factor theorem:

If $(x-a)$ is a factor of $f(x)$ then $f(a)=0$ If $f(a)=0$ then $(x-a)$ is a factor of $f(x)$	and	If $(ax-b)$ is a factor of $f(x)$ then $f(^b/_a)=0$ If $f(^b/_a)=0$ then $(ax-b)$ is a factor of $f(x)$

AS level - OCR students. A level - all students.

9. Find k if $(x-2)$ is a factor of x^3+kx^2-5x-1 Complete: Let $f(x)=x^3+kx^2-5x-1$ then $f(2)=$	*$f(2)=0$* *Substitute $x=2$. The answer $=0$ hence find k.*
10. Factorise completely the expression x^3+2x^2-5x-6 Hence sketch the curve $y=x^3+2x^2-5x-6$ showing where it cuts/touches the axes. Complete: Let $f(x)=x^3+2x^2-5x-6$ $f(1)=$ $f(2)=$ therefore ( ) is a factor. $f(\quad)=$ $f(\quad)=$ $f(\quad)=$ $f(\quad)=$ $\therefore\ x^3+2x^2-5x-6=(\quad)(\quad)(\quad)$	*Factorise.* *Try $x=1, 2, 3, -1, -2$ etc.* *When you find a factor write this down. Once you have found one factor for a cubic you can search for other factors in the same way, or write the cubic as a product of one linear, and one quadratic factor, and then factorise the quadratic factor if possible.* *Here we are searching.* *Once you have found two factors the third is easy to find: the product of the three numbers in the brackets must give -6 in this example.* *Sketch the curve showing where it cuts/touches the axes.*
11. Write x^3-x^2-5x+6 as the product of one linear factor and one quadratic factor. Complete: Let $f(x)=x^3-x^2-5x+6$ $f(1)=$ $f(2)=$ $\therefore\ x^3-x^2-5x+6=(\quad)(\quad)$ Hence solve the equation $x^3-x^2-5x+6=0$ giving each root in exact form.	*Search for the linear factor. Try $x=1, 2, 3, -1, -2$ etc.* *Then to find the quadratic.* *Either:* • *Write = (your factor) (ax^2+bx+c) and expand the brackets and equate coefficients.* • *Write with the x^2 and the constant terms and work out the coefficient of x by considering the terms that multiply to give the x^2, or the x term in the cubic.* • *Divide.* *Write with the found factors. Try to factorise the quadratic. If it does not factorise you will have to use the quadratic formula to find the other two roots (the clue here is the question asks for exact form - so perhaps you would realise the formula would be required). Leave these two answers in surd form. Don't forget the root from the linear bracket.*

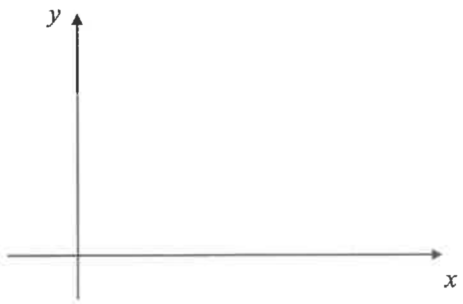
12. Solve the equation $x^3 - 3x^2 - 13x + 15 = 0$	*First **factorise** as far as possible.* *Write out the equation with the factors.* *Hence the solutions for x.*
13. Show that $(2x - 1)$ is a factor of $2x^3 - x^2 - 2x + 1$	*AS level - OCR students.* *A level - all students.*

COORDINATE GEOMETRY

The circle:

A circle with centre (a , b) and radius r has equation:

$$\boxed{(x-a)^2 + (y-b)^2 = r^2}$$

14. Find the equation of a circle centre (3 , 5) and radius 4. ( ) + ( ) =	*Use formula.*
Sketch the circle.  Write the equation in the form $x^2 + y^2 + px + qy + c = 0$	*Multiply out the brackets and tidy up.* *Remember to work out (x–3)(x–3) when squaring the first bracket and so on.*

15. Write down the equation of a circle with centre (0 , 4) and radius 5. Where does this circle intersect the x axis?

16. Find the points of intersection of the circle $(x-4)^2 + (y-2)^2 = 25$ and the y axis.

17. AB is the diameter of a circle. If the coordinates of A and B are $(3, 7)$ and $(5, 3)$ respectively, find the equation of the circle.

To find the centre and radius of a circle given in the form $x^2 + y^2 + px + qy + c = 0$ you can complete the square on xs and ys to obtain the form $(x - a)^2 + (y - b)^2 = r^2$

Example: Find the centre and radius of the circle given by $x^2 + y^2 + 6x - 10y - 2 = 0$

$x^2 + 6x + y^2 - 10y - 2 = 0$	Rewrite with the xs and ys together.
$(x + 3)^2 - 9 + (y - 5)^2 - 25 - 2 = 0$	Complete the square on xs and ys.
$(x + 3)^2 + (y - 5)^2 = 36$	Tidy up.

i.e. a circle centre $(-3, 5)$ and radius 6

18. Find the centre and radius of the circle given by $x^2 + y^2 + 8x + 3y - 2 = 0$

Reminder - circle properties:

- The angle in a semicircle is a right angle.
- A perpendicular from the centre to a chord bisects the chord.
- A radius and associated tangent are at right angles.

19. Find the mid-point of the line joining $A(2, 12)$ and $B(10, 8)$
    Find the equation of the perpendicular bisector of the line AB.
    This perpendicular bisector cuts the x axis at C. Find C.
    Find the equation of a circle with centre C which passes through the points A and B.
    Write this equation in the form $x^2 + y^2 + px + qy + c = 0$

20. Find the coordinates of the points of intersection of the circle $(x-1)^2+(y-2)^2=25$ and the line $y+x=8$	*Rearrange the linear equation to give y in terms of x (or x in terms of y).* *Substitute into the equation of the circle.* *Expand brackets and tidy up.* *Solve the resulting equation by factorising, or using the quadratic formula.* *Find the corresponding values of y.* *Question asks for the points, so write them down.*
21. Show that the line $y=x+3$ is a tangent to the circle $x^2+y^2-8x-6y+17=0$	*Substitute $y = x + 3$ into the circle equation. If the line touches the circle there will be two equal roots for the solution of the resulting quadratic equation.*

22. The coordinates of P, Q and R are $(1, 2)$, $(9, -4)$ and $(21, 12)$ respectively. Find the length PQ. Show that PQR is a right angle. Hence find the equation of the circle with centre P and has QR extended as a tangent.

EXPONENTIALS AND LOGARITHMS

Logarithms:

Logarithm is another name for power.

We know $10^0 = 1$ and $10^1 = 10$ and $10^{½} = 10^{0.5} = \sqrt{10} \approx 3.1623$

Then if $10^x = 3$, x must be about 0.4

x is called the logarithm of 3 to the base 10. x is approximately 0.4771

We can write $10^x = 3$ or $\log_{10} 3 = x$ $10^{0.4771} = 3$ or $\log_{10} 3 = 0.4771$

In general $\boxed{y = \log_a x \iff a^y = x}$

Perhaps easier to remember a specific case and generalise from this.
e.g. $10^2 = 100 \iff \log_{10} 100 = 2$ Remember: log is a power.

23. From your knowledge of powers of 10, complete: $\log_{10} 10 = 1$ since $10^1 = 10$ $\log_{10} 100 = 2$ since __________ $\log_{10} 1{,}000{,}000 =$ __________ since $10^6 = 1{,}000{,}000$ $\log_{10} 1 =$ ___ since __________ $\log_{10} 0.01 =$ ___ since __________	$0.01 = \dfrac{1}{100} = 10^{-2}$
24. Complete: $\log_3 9 =$ _______ since $3^2 = 9$ $\log_{16} 4 =$ _______ since $16^{½} = \sqrt{16} = 4$	

The laws of logarithms:

Law 1 $\log_c ab = \log_c a + \log_c b$

Law 2 $\log_c \dfrac{a}{b} = \log_c a - \log_c b$

Law 3 $\log_c a^n = n \log_c a$

Also:
$\log_c 1 = 0$
$\log_c c = 1$

Be aware that these log laws and outcomes can be used both ways.
i.e. 0 can be written as $\log_{10} 1$
and 1 can be written $\log_{10} 10$
$\log_{10} a + \log_{10} b = \log_{10} ab$
etc.

These laws help you to solve most problems involving powers and logarithms.

From law 2, $\log_c \frac{1}{n} = \log_c 1 - \log_c n$

Since $c^0 = 1$, $\log_c 1 = 0$ Hence, for any base c, $\log_c \frac{1}{n} = -\log_c n$

This means that for $\log_c n$
if n = 1, its logarithm is zero
if n > 1, its logarithm is positive
and if 0 < n < 1, its logarithm is negative
and if n < 0 the logarithm does not exist - You cannot have the logarithm of a negative number.

CHECK THESE OUT USING YOUR CALCULATOR

Useful to know:

If $y = a^{\log_a f(x)}$

then $\log_a y = \log_a a^{\log_a f(x)}$

$\log_a y = \log_a f(x) \log_a a$

$\log_a y = \log_a f(x)$

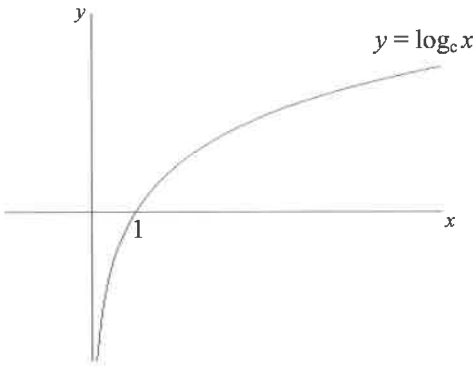
$y = f(x)$

So $a^{\log_a f(x)} = f(x)$

Graphs of logarithms:

Here is the graph of $y = \log_c x$

All log graphs look like this, whatever the base.



Note:
- The y axis is an asymptote.
- The graph cuts the x axis at $x = 1$.
- The graph increases all the time! (There is no horizontal asymptote.)

Solving equations where the unknown is a power:

You will need to take logs and use the log laws $\log_c ab = \log_c a + \log_c b$ and $\log_b a^n = n \log_b a$

Best by example: Solve $2^{3x} = 5$

Take logs of both sides $\log_{10} 2^{3x} = \log_{10} 5$

Use a log law $3x \log_{10} 2 = \log_{10} 5$

Divide by log 2 $3x = \frac{\log 5}{\log 2} = \frac{0.6989700043}{0.3010299957} = 2.321928095$

Divide by 3 $x = 0.773976 \quad = 0.774$ to 3 sig. fig.

Example: Solve $5000 \times 2^{-4t} = 200$

Divide by 5000 $2^{-4t} = 0.04$

Take $\log_{10}$ $\log_{10} 2^{-4t} = \log_{10} 0.04$

Use log law $-4t \log_{10} 2 = \log_{10} 0.04$

$-4t = \log_{10} 0.04 \div \log_{10} 2$

Divide by -4 $t = (-4.64385619) \div (-4)$

$= 1.16$ to 3 sig. fig.

> Note: If you take logs first you can solve the equation as shown.
>
> Take log of both sides $\log 5000 \times 2^{-4t} = \log 200$ (power is only with the 2)
>
> Use a log law $\log 5000 + \log 2^{-4t} = \log 200$
>
> Use a log law $\log 5000 + -4t \log 2 = \log 200$
>
> $\log 5000 - 4t \log 2 = \log 200$
>
> $-4t \log 2 = \log 200 - \log 5000$
>
> $-4t = (\log 200 - \log 5000) \div \log 2$
>
> $= -4.64385619$
>
> $t = 1.16$ to 3 sig. fig.

Complete these examples on logarithms:

25. Solve the equation $5^n = 60$ $\log_{10} 5^n = \log_{10} 60$ Using law 3 $\quad = \log_{10} 60$ Using a calculator, $\quad n = \quad =$ and hence $\quad n = \quad$ (working to 5 decimal places)	*Take logarithms base 10 on both sides and use the laws of logarithms to simplify the problem.* *$\log_{10}$ is log on your calculator.* *Check this answer by finding the value of $5^{2.54396}$*
26. If $\log_x 2 = 0.6$ find $\log_x 0.5$ $0.5 = \frac{1}{2}$ so $\quad \log_x 0.5 = \log_x \frac{1}{2} = \log_x \quad - \log_x$ $= \quad =$	*Use $\log_c \frac{1}{n} = \log_c 1 - \log_c n = -\log_c n$* *All logarithms of numbers between 0 and 1 are negative. See NOTE above.* *$\log_x 1 = 0$.*

27. Write $2 \log 3 + \frac{1}{2} \log 25 - \log 4 + \log \frac{1}{3}$ as a single logarithm, assuming all logs are base 10.

Complete: $2 \log 3 = \log 3^2 = \log 9$, $\quad \frac{1}{2} \log 25 = \log \quad = \log \quad$ and $\log \frac{1}{3} = -\log$

So $2 \log 3 + \frac{1}{2} \log 25 - \log 4 + \log \frac{1}{3} = \log 9 + \log 5 - \log 4 - \log 3 = \log$

$= \log$

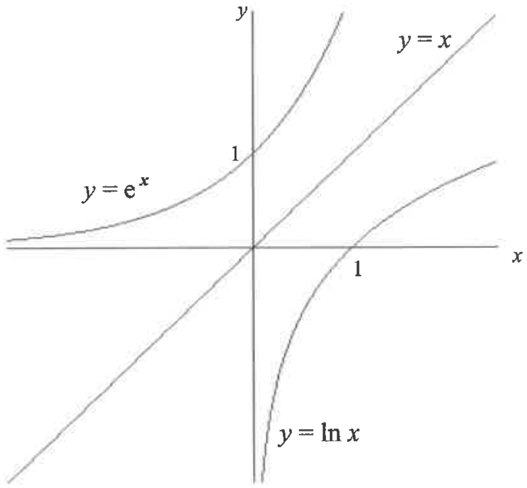
$= \log$

28. If a Building Society offers a savings scheme with compound interest at 5% per annum, how long would it take an initial investment of £200 (never added to) to grow to £300? Assuming the interest is added at the end of each year: At the end of year 1 the total is $\frac{105}{100} \times £200$, or $1.05 \times £200$ At the end of the year 2 the total is $1.05\,(1.05 \times £200) = 1.05^2 \times £200$ and so on. We want to know the number of years, n, where $1.05^n \times £200 = £300$ Complete: $1.05^n =$ $\log 1.05^n = \log 1.5$ $= \log 1.5$ $n = \qquad = \qquad$ (2 d.p.) so the money needs to be left for 9 years before the total is greater than £300.	*Without doing the calculations work out the amount at the end of each year. This way you will see a pattern.* *Dividing both sides by 200.* *Taking logs of both sides.* *Use the law to bring that unknown power down.* *Dividing both sides by log 1.05 to give n.*
29. Express $\log_{10} 47 = 1.672$ in the form as a power of 10.	*Remember: log is a power and if $10^2 = 100$ then $\log_{10} 100 = 2$*
30. Evaluate the following without using your calculator: $\log_{10} 14 + \log_{10} 5 - \log_{10} 7$	*Use the laws to simplify.*
31. Simplify $\log_{10} x^3 + \log_{10} x$	*Use the laws to simplify.*
32. Evaluate $\dfrac{\log x^3}{\log x}$	*Use a law to change the top.* *(This is NOT $\log \frac{a}{b}$)*
33. Find n if $2 \times 3^n = 500$ Give your answer to 3 significant figures.	*Easiest to first divide by 2.* *Take logs of both sides.* *Then use a law to bring the unknown down so it is not a power.* *Find n..*

34. Solve the equation $3 \log_a x = \log_a 125$

35. Given that $\log_a y = \log_a 3 + 2$, express y in terms of a, giving your answer in a form not involving logarithms.

The exponential function and natural logarithms:



e^x is *the exponential function* - where $e = 2.718\ldots$

ln is $\log_e$

e^x and $\ln x$ are inverse functions.

All the previous work done on indices and logarithms apply:

If $\ln x = a$ then $x = e^a$	Definition
$\ln a + \ln b = \ln ab$ $\ln a - \ln b = \ln \frac{a}{b}$ $\ln a^b = b \ln a$	Laws of logs

Also note: $\ln 1 = 0$
$\ln e = 1$
$\ln e^x = x \ln e = x$

The gradient function of e^x is e^x and the gradient function of e^{kx} is ke^{kx} for any k.
When the rate of change of y is proportional to the y value, an exponential model should be used.
For example: $y = Ae^{kx}$ or $P = P_0 e^{0.05t}$ etc. for growth, and $y = Ae^{-kx}$ or $m = m_0 e^{-kt}$ etc. for decay.

36. Evaluate the following: (a) $\ln 3$ (b) $\ln 13.5$ (c) e^3 (d) $e^{1.6}$	*Just use your calculator.*
37. Express the following as a sum or difference of logarithms: (a) $\ln \dfrac{x}{2x+1}$	*Use the laws of logs to rewrite.*
(b) $\ln 8x^2$	*Use ln ab law <u>then</u> ln a^b law.*
(c) $\ln \tan x$	$\tan x = \dfrac{\sin x}{\cos x}$
38. Express the following as a single logarithm: (a) $\ln(2x+1) + \ln A$	*Use the laws of logs to rewrite.* *(a) ln ab*
(b) $1 - \ln x$	*(b) Rewrite 1.*
(c) $2\ln x + \frac{1}{2}\ln(x+1)$	*(c) Use ln a^b law first <u>then</u> ln ab law.*

Solving equations: Give your answers to 3 significant figures unless told otherwise.

Example: Solve $2e^{5x+3} = 5$ $e^{5x+3} = 5/2$ $\ln e^{5x+3} = \ln(5/2)$ $(5x+3)\ln e = \ln(5/2)$ $(5x+3) = \ln(5/2)$ $x = [\ln(5/2) - 3] \div 5$ $= -0.417$ to 3 sig.fig.	*Divide by 2* *Take logs base e* *Use log law* *ln e = 1* *− 3 and ÷ 5.* *Line 4 could be obtained direct using definition.*	Example: Solve $2\ln 5x = 1$ $\ln 5x = ½$ $5x = e^{½}$ $x = e^{½} \div 5$ $= 0.330$ to 3 sig.fig.	*÷2* *Use definition* *÷ 5*

39. Solve for x

(a) $e^{x} = 5.4$ (b) $e^{2x+1} = 10$

(c) $e^{2x} + e^{x} - 6 = 0$

Complete: $(e^{x})^2 + (e^{x}) - 6 =$

$=$

$(\quad)(\quad) =$

$e^{x} =$ or $e^{x} =$

(a) Take logs (base e) of both sides.
Remember ln e = 1
OR just use the definition.
(b) Take logs and then bring down the power.
OR just use the definition.
(c) $e^{2x} = (e^{x})^2$ so this is a quadratic in e^{x}.
You may wish to rewrite, substituting y for e^{x}.
Factorise the quadratic.

Take logs of both and find x.

Write out your solution.

40. Sketch the graphs:

$y = -e^{x}$ $y = e^{-x}$ $y = -e^{-x}$

Careful here. It is easy to confuse e^{-x} and $-e^{x}$. Easiest to sketch e^{x} lightly first.

41. Solve the equation $5^{x} = 100$

Take ln of both sides.
Use log law.
Divide by ln 5.

42. Solve the equation $\ln 2x = 0.5$

43. Solve the equations (a) $e^{x} = 4$ (b) $3^{x+2} = 86$

44. If $P = 2000\,e^{-0.2T}$ find T when $P = 500$

Write equation with 500 for P.

Divide by 2000.

Take ln of both sides.

Use log law and ln e = 1.
Divide by – 0.02 to give T.

45. If $S = R\,e^{kt}$ find k, to 3 decimal places, given $S = 1500$ and $R = 1200$ when $t = 5$.

Write equation with known values.

Divide by 1200.

Take ln of both sides.

Use log law and ln e = 1.

Divide by 5 to give k.

46. If $m = m_0 e^{-kt}$ find t when $m = 1$ given $m_0 = 2$ and $k = 0.02$	*Write equation with known values.* *Divide by 2.* *Take ln of both sides.* *Use log law and ln e = 1.* *Divide by – 0.02 to give k.*
47. Given that $y = \mathrm{A}\,e^{kt}$ and $y = 15$ when $t = 0$, and $y = 60$ when $t = 1$, find y when $t = 2$	*y = 15 when t = 0. Substitute values. (remember* e^0 *=?)* *y = 60 when t = 1. Substitute.* *Take ln of both sides.* *Tidy up to find k.* *Rewrite the equation with the values of A and k.* *Now find y when t = 2*
48. A substance decays at a rate given by $\mathrm{m} = \mathrm{A}\,e^{-kt}$, where m is the mass in grams after t days and A and k are constants. Initially m = 10. If m = 9 grams when $t = 4$ days, find the value of k. Also find the mass after one week.	*Could be worded - The rate of decay of a substance at time t, is proportional to m, the mass of the substance, at time t. You would then write down* $\mathrm{m} = \mathrm{A}\,e^{-kt}$ *Here A and k are used but you could use any constants.* *Initially means when t = 0. m = 10 when t = 0 gives the value of A. Substitute into formula.* *Now find k by substituting m = 9 and t = 4 into the formula.* *Write down the formula with the known constant.* *Now find m when t = 7*

Reduction to linear form:

A relationship of the form $y = ax^n$ can be reduced to a linear form by taking logarithms, since

$$y = ax^n$$

Take logs of both sides (any base) $\log y = \log(ax^n)$

Use log law (only x is to the power n) $\log y = \log a + \log(x^n)$

Use log law $\log y = \log a + n\log x$

So $\log y = n\log x + \log a$ Note: log any base can be used.

Compare $\log y = n\log x + \log a$

with $\mathrm{Y} = \mathrm{m}\,\mathrm{X} + \mathrm{c}$ you can see that plotting values of $\log y$ against values of $\log x$ gives a straight line with gradient n and vertical intercept $\log a$

A relationship of the form $y = kb^x$ can be reduced to a linear form by taking logarithms, since

$$y = kb^x$$

Take logs of both sides (any base) $\log y = \log(kb^x)$

Use log law (only b is to the power x) $\log y = \log k + \log(b^x)$

Use log law $\log y = \log k + x\log b$

So $\log y = (\log b)x + \log k$ Note: log any base can be used.

Compare $\log y = (\log b)x + \log k$

with $\mathrm{Y} = \mathrm{m}\ \mathrm{X} + \mathrm{c}$ you can see that plotting values of $\log y$ against values of x gives a straight line with gradient $\log b$ and vertical intercept $\log k$

Note: When plotting your values they may not lie exactly on a straight line. As long as they are close then you can accept that within small experimental/reading error the points line on a straight line.

49. The variables x and y are known to be related by an equation in the form $y = ab^{x}$ where a and b are constants.

The following approximate values of x and y have been found.

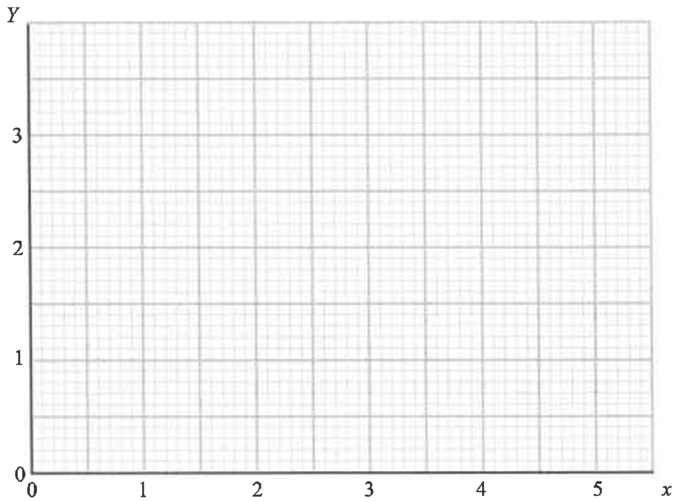
x	1	2	3	5
y	63.25	145.5	334.6	1770.0

(a) Complete the table, showing values of x and Y, where $Y = \log_{10} y$. Give each value of Y to three decimal places.

x	1	2	3	5
Y				

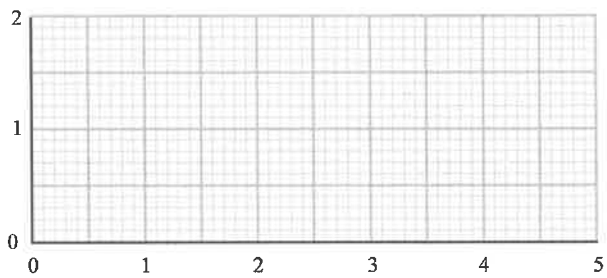
(b) Show that, if $y = ab^{x}$, then x and Y must satisfy an equation of the form $Y = mx + c$

(c) Draw a graph relating x and Y. Hence find estimates for the values of a and b



50. The variables x and t are believed to satisfy a relationship of the form $t = kx^{n}$ where k and n are constants. Show that the experimental values given in the table do satisfy the relationship using the graph. Find approximate values for k and n.

x	10	100	1000	31623
t	3.55	7.59	16.6	52.7



Mixed questions 1

51. Solve the equation $3^{x} = 77$ Give your answer to 3 significant figures.

52. Factorise completely the expression $x^{3} + 4x^{2} + x - 6$

53. A circle has centre $(1, -4)$ and radius 5. Find the equation of the circle in the form $x^{2} + y^{2} + px + qy + c = 0$ Also find where this circle cuts the x axis.

54. The population of a certain area grows according to the law $P = A\,e^{0.05t}$ where P is the population at time t years and A is a constant. When $t = 0$ the population is 3.4 million.
Calculate the time when the population has doubled.

Could be worded - The rate of growth of a population at time t, is proportional to P, the size of the population at time t. You would then write down $P = A\,e^{kt}$ *Here A and k are used but you could use any constants. Other information would then be given to enable you to find k.*

55. Solve the equation $\log_{10} 5x = 1.5$ Give your answer to 3 significant figures.

56. Solve the equation $10^{x} = 4$ Give your answer to 3 significant figures.

57. Solve the equation $5^{x+3} = 123$ Give your answer to 3 significant figures.

58. Write $x^3 + x^2 - 2x + 12$ as the product of one linear and quadratic expression.

59. Solve the equation $\log_{10} x + 3\log_{10} 2 = 2$

60. Solve the equation $2^{3x} - 34 = 0$ Give your answer to 3 significant figures.

61. The variables r and t are believed to satisfy a relationship of the form $t = kr^n$ where k and n are constants. Show that the experimental values given in the table do satisfy the relationship using the graph. Find approximate values for k and n.

r	1	2	3	4	5
t	0.6	0.74	0.83	0.91	0.97

62. Solve the equation $2^{2x+3} - 17 \times (2^x) + 2 = 0$

63. Show that $(x - 5)$ is a factor of $x^3 - 6x^2 + 3x + 10$ and find the other linear factors. Solve the equation $x^3 - 6x^2 + 3x + 10 = 0$

SEQUENCES and SERIES

Binomial expansions:

Using Pascal's triangle to find coefficients.

1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

Note the pattern of *a*s and *x*s

Using 1 3 3 1 for coefficients $(a+x)^3 = 1a^3 + 3a^2x + 3ax^2 + 1x^3$
Using 1 4 6 4 1 for coefficients $(a+x)^4 = 1a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + 1x^4$

Expand the following:

64. $(3+x)^4 = 3^4 + 4 \times 3^3 x +$ $=$	*Complete using the numbers from the fourth row of Pascal's triangle and the pattern for 3s and xs.* *Work out the coefficients.*
65. $(1+x)^6 =$	*Work out the sixth row of Pascal's triangle.*
66. $(2-x)^5 =$	*Think of this as $(2 + (-x))^5$. Use the numbers from the fifth row of Pascal's triangle. Write the (–x) in brackets – then you will be less likely to make a mistake when tidying up your answer.*

Remember: $^5C_3 = \binom{5}{3} = \dfrac{5!}{3!(5-3)!} = \dfrac{5!}{3!2!} = \dfrac{5\times4\times3}{3\times2\times1} = 10$

Note: Quick method - Write out the bottom first (3! = 3×2×1) and then starting from 5, going down, write the same number of numbers on the top. (Also see example.)

Learn the method for expanding a bracket with two terms (a binomial expression).
For example with a power 5

$$(a+b)^5 = {}^5C_0\,a^5 + {}^5C_1\,a^4b + {}^5C_2\,a^3b^2 + {}^5C_3\,a^2b^3 + {}^5C_4\,ab^4 + {}^5C_5\,b^5$$
$$= a^5 + \frac{5}{1}a^4b + \frac{5.4}{2.1}a^3b^2 + \frac{5.4.3}{3.2.1}a^2b^3 + \frac{5.4.3.2}{4.3.2.1}ab^4 + \frac{5.4.3.2.1}{5.4.3.2.1}b^5$$
$$= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

Sometimes written

$$(a+b)^5 = \binom{5}{0}a^5 + \binom{5}{1}a^4b + \binom{5}{2}a^3b^2 + \binom{5}{3}a^2b^3 + \binom{5}{4}ab^4 + \binom{5}{5}b^5$$
$$= a^5 + \frac{5}{1}a^4b + \frac{5.4}{2.1}a^3b^2 + \frac{5.4.3}{3.2.1}a^2b^3 + \frac{5.4.3.2}{4.3.2.1}ab^4 + \frac{5.4.3.2.1}{5.4.3.2.1}b^5$$
$$= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

67. Complete for power 7
$(a+b)^7 = {}^7C_0\,a^7 + {}^7C_1\,a^6b + {}^7C_2\,a^5b^2 + {}^7C_3\,a^4b^3 + {}^7C_4\,a^3b^4 + {}^7C_5\,a^2b^5 + {}^7C_6\,ab^4 + {}^7C_7\,b^5$

$=$

$$(a+b)^n = {}^nC_0\,a^n + {}^nC_1\,a^{n-1}b + {}^nC_2\,a^{n-2}b^2 + {}^nC_3\,a^{n-3}b^3 + \ldots\ldots$$

68. Find the first three terms in the expansion of $(2a+3b)^6$	*Put brackets round (2a) and (3b) then you will be less likely to make mistakes when raising to a power.*
69. Find the first four terms in the expansion of $(p-2q)^5$ written in decreasing powers of p.	*Rewrite with a plus and bracket (– 2q) This is so that you can now use the expansion for $(a+b)^n$*

70. Expand $(2y-3)^3$

71. Expand $(1+x)^6$ up to the term in x^3

72. Expand $(1-2x)^8$ up to the term in x^3	*Rewrite in the form $(1+x)^n$* *Leave in the brackets round the (– 2x)* *Careful when multiplying out the brackets.* *Tidy up.*

73. Write down the first three terms of the expansion of $(1+3x)^7$ in ascending powers of x

74. Expand $(2+x^2)^5$

75. Expand $(1+x)(2-3x)^3$

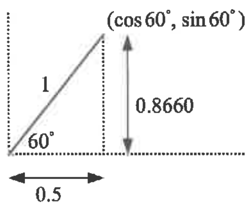
76. Expand $\left(1 - \frac{x}{2}\right)^6$

77. Expand $(1 - 5x)^7$ as far as the term in x^2

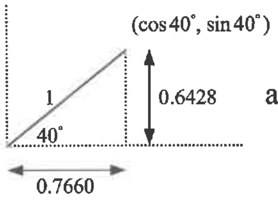
TRIGONOMETRY

Definition of sine, cosine and tangent:

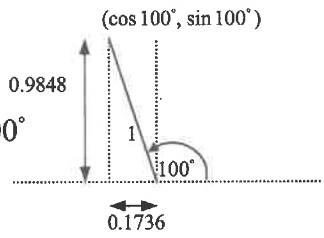
The end of a line from the origin of length 1 unit, at an angle of 60° to the x axis has coordinates $(0.5, 0.866)$
These coordinates are given by $(\cos 60^\circ, \sin 60^\circ)$



and for angle 40°

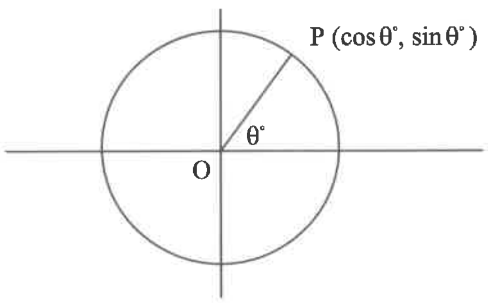


and for angle 100°



so x coordinate is -0.1736

The locus of a point P which is 1 unit from the origin is a circle radius 1 unit centre the origin.
For the line OP at an angle of θ° to the positive x axis, P has coordinates $(\cos\theta^\circ, \sin\theta^\circ)$



Angles measured **anticlockwise** from the positive x direction are **positive**.

Angles measured **clockwise** from the positive x direction are **negative**.

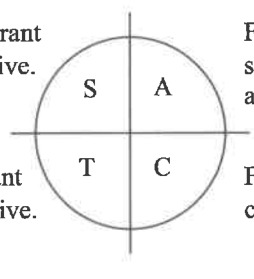
Also $\boxed{\tan\theta = \frac{\sin\theta}{\cos\theta}}$

For θ from **0° to 90°** the x coordinate is positive, so **$\cos\theta$ is positive** and the y coordinate is positive, so **$\sin\theta$ is positive** and therefore **$\tan\theta$ is positive**.
For θ from **90° to 180°** the x coordinate is negative, so **$\cos\theta$ is negative** and the y coordinate is positive, so **$\sin\theta$ is positive** and therefore **$\tan\theta$ is negative**.
For θ from **180° to 270°** the x coordinate is negative, so **$\cos\theta$ is negative** and the y coordinate is negative, so **$\sin\theta$ is negative** and therefore **$\tan\theta$ is positive**.
For θ from **270° to 360°** the x coordinate is positive, so **$\cos\theta$ is positive** and the y coordinate is negative, so **$\sin\theta$ is negative** and therefore **$\tan\theta$ is negative**.
And so on for angles greater than 360° You can also appreciate these points by drawing the graphs of sine, cosine and tan.

The A(all), S, T and C stand for the trig ratios which are positive in these quadrants.

Second quadrant
$\sin\theta$ is positive.

First quadrant
$\sin\theta$, $\cos\theta$ and $\tan\theta$ are positive.



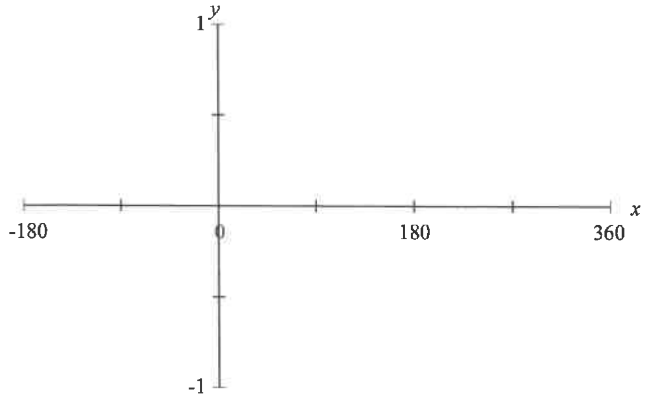
Third quadrant
$\tan\theta$ is positive.

Fourth quadrant
$\cos\theta$ is positive.

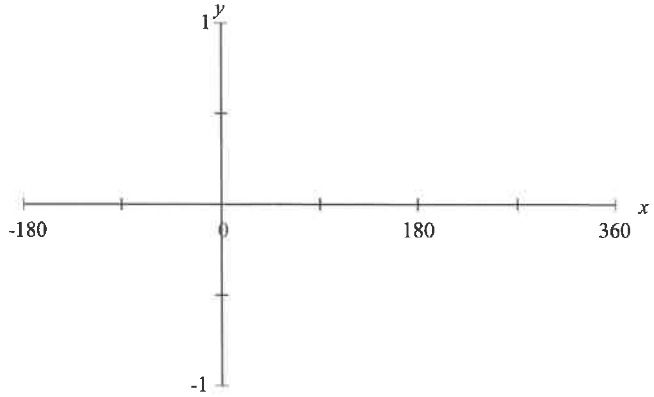
sine, cosine and tangent graphs:

Sketch the graphs.

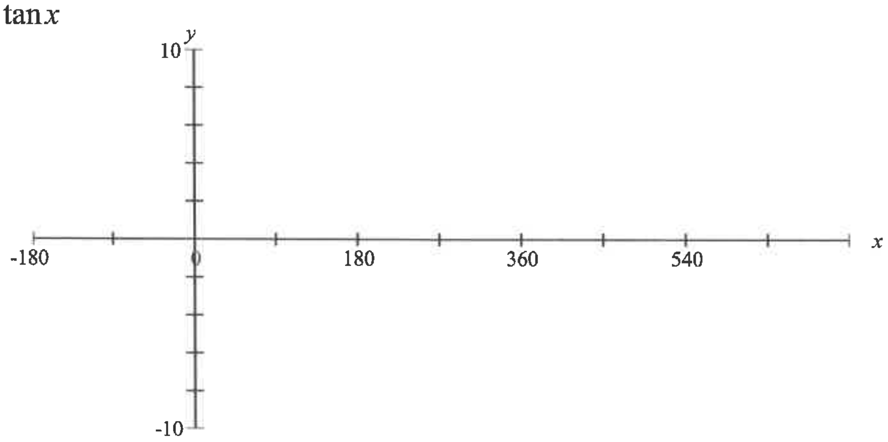
$y = \sin x$



$y = \cos x$



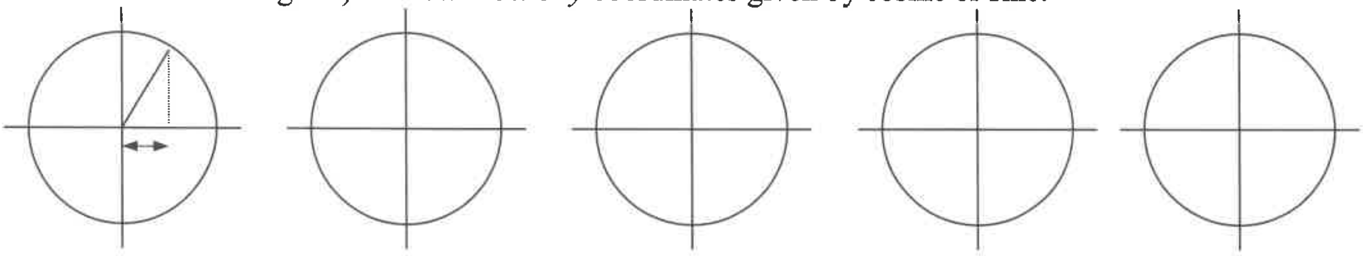
$y = \tan x$



Check these on a graphics calculator or computer.

sine, cosine and tangent of all angles:

On the following diagrams draw in the lines at the angle given (approximately) and use a coloured pen (arrowed in the first diagram) to show the x or y coordinates given by cosine or sine.



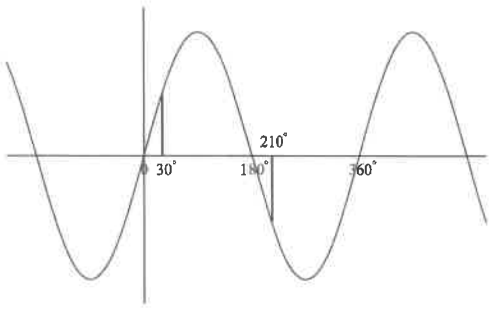
$\cos 60^\circ$ $\sin 30^\circ$ $\cos 120^\circ$ $\cos(-60^\circ)$ $\sin 150^\circ$

From similar triangles you can see that $\cos 60^\circ = \sin 30^\circ = -\cos 120^\circ = \cos(-60^\circ) = \sin 150^\circ$

Notice that $\cos 120^\circ$ is negative (as the x coordinate is negative) so to make it equal to $\cos 60^\circ$ you put a minus in front. Could also be written $\cos 120^\circ = -\cos 60^\circ$

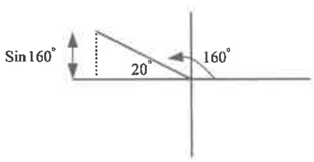
OR you can find equivalent trig ratios by looking at their graphs.

By symmetry you can see that $\sin 210^\circ$ is the same as $\sin 30^\circ$ except that $\sin 210^\circ$ is negative and $\sin 30^\circ$ is positive.
So $\sin 210^\circ = -\sin 30^\circ$

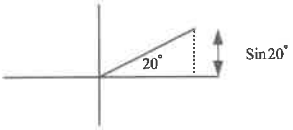


All trigonometric ratios can be expressed in terms of trigonometric ratios of acute angles.

Examples: For $\sin 160^\circ$

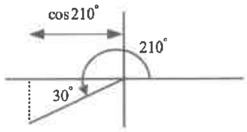


First sketch and work out angle with x axis. Then sketch a triangle with this angle in the first quadrant.

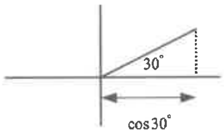


By similar triangles the y coordinate $\sin 160^\circ$ is the same as the y coordinate $\sin 20^\circ$ and they are both positive so $\sin 160^\circ = \sin 20^\circ$

For $\cos 210^\circ$



First sketch and work out angle with x axis. Then sketch a triangle with this angle in the first quadrant.



the x coordinate $\cos 210^\circ$ is negative

the x coordinate $\cos 30^\circ$ is positive

so by considering similar triangles and the sign $\cos 210^\circ = -\cos 30^\circ$

OR you can find equivalent trig ratios by looking at their graphs as above.

You can always check on your calculator that these trig ratios are equal.

78. Express the following in terms of trigonometric ratios of acute angles.

(a) $\cos 135^\circ$

(b) $\sin 135^\circ$

(c) $\cos 280^\circ$

(d) $\sin 220^\circ$

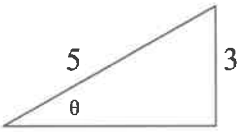
(e) $\sin 315^\circ$

(f) $\cos 560^\circ$

First sketch and work out angle with x axis. Then sketch a triangle with this angle in the first quadrant
OR
Use the symmetry of the relevant trig graph.

Opposite side, adjacent side and hypotenuse can still be used as normal.
Given one trig ratio as a fraction you can work out the other trig ratios.
Complete the following:

79. Given $\sin\theta = \frac{3}{5}$ and θ is acute.

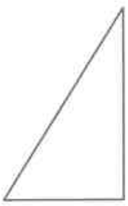


First draw a right angled triangle with θ and the two known sides. Use Pythagoras to work out the other side. Then you can work out the other trig ratios.

$\cos\theta =$ $\tan\theta =$

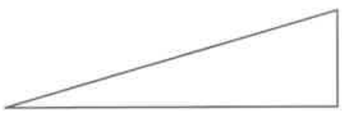
Although the angle could be worked out and cosine and tangent could be found from the angle, the ratios will generally be approximations. Use fractions to obtain the exact values of the ratios.

80. $\cos\theta = \frac{5}{13}$ and θ is acute.



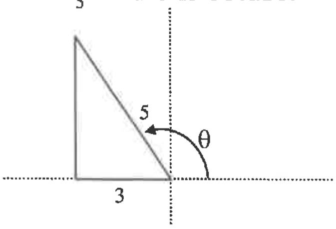
$\sin\theta =$ $\tan\theta =$

81. $\tan\theta = \frac{7}{24}$ and θ is acute.



$\sin\theta =$ $\cos\theta =$

82. $\cos\theta = -\frac{3}{5}$ and θ is obtuse.



Use Pythagoras to work out the other side. Then you can work out the other trig ratios. **Don't forget to check if they should be negative.**

$\sin\theta =$ $\tan\theta =$

83. $\sin\theta = \frac{8}{17}$ and θ is acute. Find the value of $\cos\theta$ and $\tan\theta$ in the form $\frac{a}{b}$, where a and b are integers.

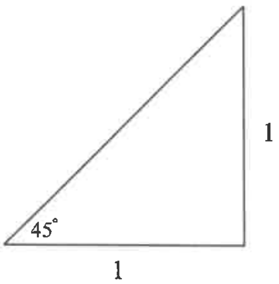
In the first quadrant. Draw a triangle with θ and the given sides. Use Pythagoras to work out the other side. Then you can work out the other trig ratios.

84. $\cos\theta = -\frac{20}{29}$ and $180^\circ < \theta < 360^\circ$ Find the value of $\sin\theta$ and $\tan\theta$ in the form $\frac{a}{b}$, where a and b are integers.

85. $\tan\theta = -\frac{3}{4}$ and $180^\circ < \theta < 360^\circ$ Find the value of $\sin\theta$ and $\cos\theta$ in the form $\frac{a}{b}$, where a and b are integers.

86. $\tan\theta = \frac{15}{8}$ and θ is acute. Find the value of $\sin\theta$, $\cos\theta$ and $\sin^2\theta + \cos^2\theta$

45° right angled triangle



Special cases - right angled triangles:

87. COMPLETE:

Find the hypotenuse $\sqrt{\qquad\qquad} = \sqrt{\quad}$

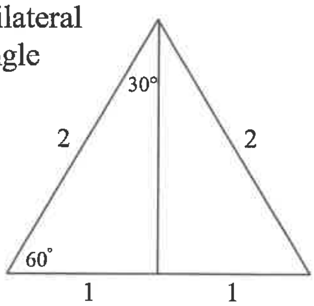
Write this in on the triangle.

Now complete these as fractions with square root signs in them.

$\sin 45^\circ =$ $\cos 45^\circ =$

$\tan 45^\circ =$

Equilateral triangle



Find the vertical height in the equilateral triangle.

$\sqrt{\qquad\qquad} = \sqrt{\quad}$

Write this in on the triangle.

Now complete these as fractions with square root signs in them.

$\sin 60^\circ =$ $\cos 60^\circ =$ $\tan 60^\circ =$

$\sin 30^\circ =$ $\cos 30^\circ =$ $\tan 30^\circ =$

You should remember how to find these trig ratios.

88. Write down the values of the following as fractions leaving surds in your answers: (a) $\cos 135^\circ$ (b) $\sin 135^\circ$ (c) $\cos 120^\circ$ (d) $\sin 210^\circ$ (e) $\tan 210^\circ$ (f) $\cos 330^\circ$	*First sketch and work out the acute angle with x axis. Then sketch a triangle with this angle in the first quadrant. This should be one of the above angles 45° 30° or 60°* *OR* *Sketch the graph and use the symmetry to work out the acute angle.* *Don't forget to check if they should be negative.*

89. Write down the values of the following:

(a) $\sin 0^\circ =$ (b) $\cos 0^\circ =$ (c) $\tan 0^\circ =$ (d) $\sin 90^\circ =$ (e) $\cos 90^\circ =$ (f) $\sin 180^\circ =$ (g) $\cos 180^\circ =$

(h) $\tan 180^\circ =$ (i) $\sin 270^\circ =$ (j) $\sin 360^\circ =$ (k) $\cos 270^\circ =$ (l) $\cos 360^\circ =$ (m) $\tan 360^\circ =$

Sketching trig graphs:

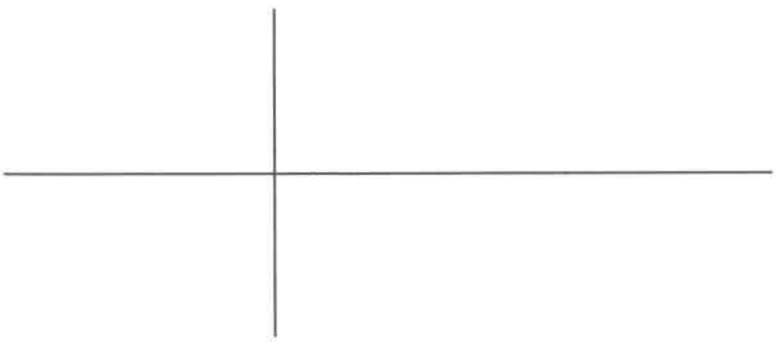
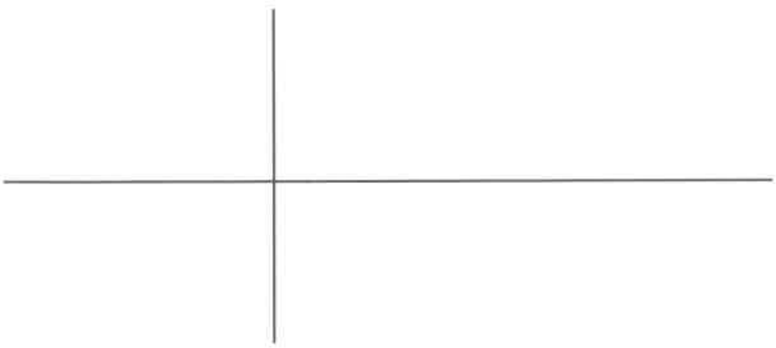
***y* = a sin (*x* + b) + c**

a will stretch $y = \sin x$ by a factor of a parallel to the y axis away from the x axis. (The amplitude.)

b will translate $y = \sin x$ by vector $\begin{bmatrix} -b \\ 0 \end{bmatrix}$ (The phase shift.) **c** will translate $y = \sin x$ by vector $\begin{bmatrix} 0 \\ c \end{bmatrix}$

y* = sin d*x

d will stretch $y = \sin x$ by a factor of 1/d parallel to the x axis away from the y axis. (The period is 360 ÷ d)

90. Sketch the curve of $y = \sin 2x$ for $-90^\circ < x < 360^\circ$ 	*Label the axes. x from − 90° to 360°* *y from − 1 to 1* *You may wish to draw y = sin x faintly to start with.* *Stretch by factor ½ parallel to the x axis away from the y axis.* *Sketch the curve. Label where the curve cuts the axes and the amplitude.*
91. Sketch the curve of $y = \cos(x - 30^\circ)$ for $-60^\circ < x < 360^\circ$ 	*Label the axes. x from − 60° to 360°* *y from −1 to 1* *You may wish to draw y = cos x faintly to start with.* *Translate 30° to the right. Easiest to mark where the curve cuts the x axis and the turning points and then join up these marks.* *Label where the curve cuts the axes and the amplitude.*

92. Sketch the curve of $y = \sin(x + 45^\circ)$ for $-60^\circ < x < 360^\circ$

Label the axes. x from -60° to 360°
y from -1 to 1

You may wish to draw y = sin x faintly to start with.

Translate 45° to the left.
Easiest to mark where the curve cuts the x axis and the turning points and then join up these marks.

Label where the curve cuts the axes and the amplitude.

93. Sketch the curve of $y = -3\cos x$ for $-90^\circ < x < 360^\circ$

Label the axes. x from -90° to 360°
y from -3 to 3

You may wish to draw y = cos x faintly to start with.

Amplitude 3. You may wish to draw y = 3cos x faintly.

Reflect in the x axis. Sketch the curve.

Label where the curve cuts the axes and the amplitude.
If you have drawn more than one curve to help you with the final sketch then clearly label the curves.

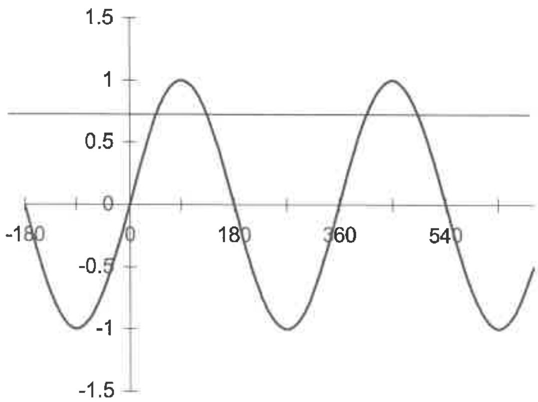
Solving trigonometric equations:

By example:

94. Solve the equation $\sin x = 0.75$ for values of x in the range $-180^\circ \le x \le 540^\circ$

$\sin x = 0.75$

$x = \arcsin 0.75 = 48.6^\circ$,........,........,........



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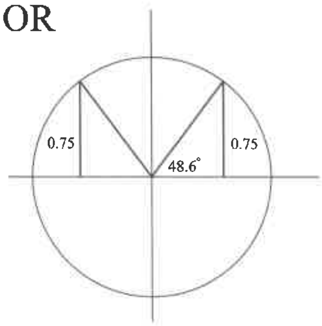


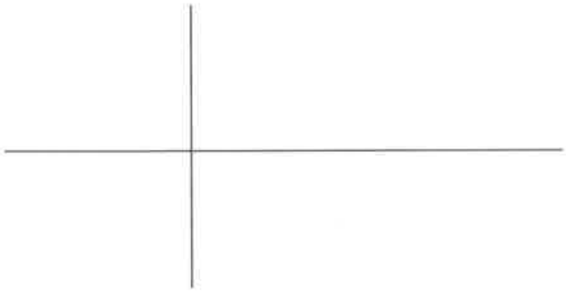
Find the principal value from your calculator (INV sin)
Range could be written [–180°,540°]

arcsin 0.75 means the angle whose sine is 0.75
Use the method you have been taught.

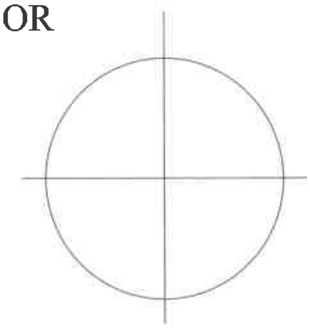
Method 1
Sketch the graph for y = sin x.
Sketch in a horizontal line at about 0.75
Draw a line down at all the points where the line crosses the curve.
The calculator gives one value of x
Use the symmetry of the curve to work out the other values of x within the range.
Write them down.

Method 2
Using the unit circle, sine (the y coordinate) is +ve in the first and second quadrants. The initial angle is 48.6, the next is 180 – 48.6, the next is 360 + 48.6 etc.

95. Solve $\cos x = -0.6$ for x in the interval $-180^\circ \le x \le 360^\circ$



OR



Complete:

$\cos x = -0.6$

$x = \arccos -0.6$

$x = \quad , \quad ,$

Range could be written [–180°,360°]
The calculator gives one value of x.
Write it down.
Use the method you have been taught.

Method 1
Sketch the curve y = cos x
Sketch in a horizontal line at about – 0.6
Draw a line down at all the points where the line crosses the curve.
Use the symmetry of the curve to work out the other values of x within the range.
Write them down.

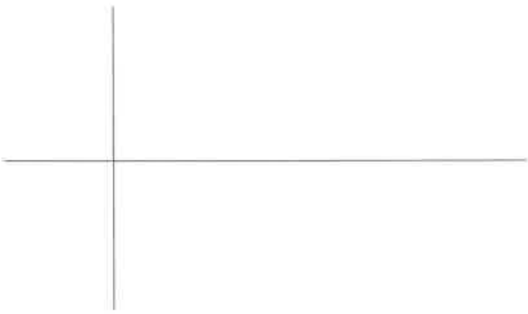
Method 2
Using the unit circle, cosine (the x coordinate) is –ve in the second and third quadrants. The initial angle from the calculator and remaining angles within the range can be found from the symmetry.

96. Solve $5\sin x = 2$ for x in the range $0^\circ < x < 720^\circ$

$\sin x =$

$x = \arcsin$

$x =$



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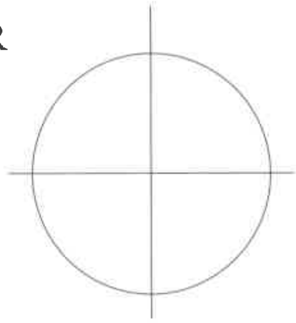


Divide by 5.

Find first value for x.
Sketch graph or unit circle.

Work out the other values of x.

97. Solve $\sin x = 0$ for x in the range $0^\circ \le x \le 720^\circ$

98. Solve $\sin 2x = 0.3$ for x in the range $0^\circ < x < 360^\circ$

Complete: $\sin A =$

$A = \arcsin \quad =$

$A = 2x =$

$x =$

sin of an angle (2x) equals 0.3
Let the angle be A.

Find first value for A.

x between 0° and 360°
Therefore A (=2x) between 0° and 720°
Sketch sin graph or use circle diagram.

Work out the other values of A up to 720°
Divide all answers by 2 to find the values of x.

99. Solve $\cos(x - 60^\circ) = 0.2$ for x in the range $0^\circ < x < 360^\circ$ Complete: $\cos A =$ $A = \arccos \qquad =$ $A = x - 60^\circ =$ $x =$	*cos of an angle (x – 60°) equals 0.2* *Let the angle be A.* *Find first value for A.* *x between 0° and 360°* *Therefore A (x – 60) between – 60° and 300°* *Sketch cos graph or use circle diagram.* *Work out the other values of A from – 60° to 300°* *Add 60° to all the answers to obtain the values of x.*
100. Solve $\tan 3x = 3$ for x in the range $0^\circ < x < 180^\circ$ Complete: $\tan A =$ $A = \arctan \qquad =$ $A = 3x =$ $x =$	*tan of an angle 3x equals 3* *Let the angle be A.* *Find first value for A.* *x between 0° and 180°* *Therefore A (3x) between 0° and 540°* *Sketch tan graph or use circle diagram.* *Work out the other values of A from 0° to 540°* *Divide all answers by 3 to find the values of x.*
101. Solve $\sin(2x + 30^\circ) = 0.5$ for x in the range $0^\circ < x < 360^\circ$	*You are going to subtract 30° and divide by 2 in your answers. You must allow for this in your first stage of answers.*

Other trigonometric functions and identities:

Remember: $\boxed{\tan\theta = \frac{\sin\theta}{\cos\theta}}$

Also: $\boxed{\sin^2 x + \cos^2 x = 1}$

Always think ‘ $\sin^2 x$ is $(\sin x)^2$ ’ and ‘ $\cos^2 x$ is $(\cos x)^2$ ’
Rearranging gives: $\sin^2 x = 1 - \cos^2 x$
and $\cos^2 x = 1 - \sin^2 x$

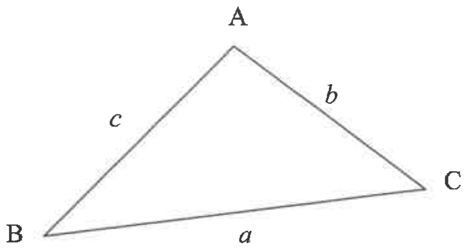
102. Solve $3\sin x + 5\cos x = 0$ for $-360^\circ \le x \le 360^\circ$ Complete: $3\sin x =$ $\frac{\sin x}{\cos x} =$ $=$ $x =$	*Range could be written [– 360˚,360˚]* *Rearrange to enable you to obtain tan x* *Here tan x =* *Your calculator gives one value for x.* *Sketch graph of tan x or the unit circle to find other values in the range.* *Give answers to one decimal place.*
103. (a) Solve the equation $2y^2 + y - 1 = 0$ (b) By rewriting $1 + \sin x - 2\cos^2 x = 0$ solve the equation for $-180^\circ \le x \le 180^\circ$ (a) Complete: $(\quad)(\quad) = 0$ $y =$ (b) $1 + \sin x - 2(\quad) = 0$ $= 0$ $= 0$ $\sin x =$ or $\sin x =$	*Factorise.* *Find y.* *Use the identity $\sin^2 x + \cos^2 x = 1$* *Multiply out the bracket and tidy up. Mind the minus sign!* *See how this compares with the equation in (a). Instead of y you have sin x to solve. The answers will be the same for y and sin x.* *For each sin x :* *Your calculator gives one value for x.* *Sketch graph of sin x or the unit circle to find other values in the range.*

The sine rule:

Easiest form to use to find a side $\boxed{\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}}$

OR

Easiest form to use to find an angle $\boxed{\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}}$



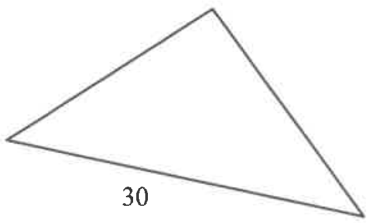
Beware of the 'ambiguous case' when trying to find an angle.

Area of a triangle:
A useful alternative.

$\boxed{\text{Area of a triangle } = \tfrac{1}{2}ab\sin C}$ *or* $\tfrac{1}{2}bc\sin A$ or $\tfrac{1}{2}ac\sin B$

104. In a triangle PQR, PQ = 30, angle P = 48° and angle Q = 34°
Calculate the lengths of the other sides.

Complete:



A sketch of the triangle is given. Label the vertices and angles.

Work out the other angle and label. (Angles in a triangle).

$\dfrac{PR}{\qquad} = \dfrac{30}{\qquad}$

Find PR. Answers to 3 sig fig.

Multiply both sides by sin 34°

$PR =$

$\dfrac{QR}{\qquad} = \dfrac{\qquad}{\qquad}$

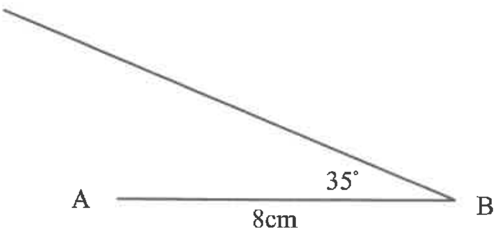
Find QR using sine rule (not Pythagoras as it is not a right angled triangle).

$QR =$

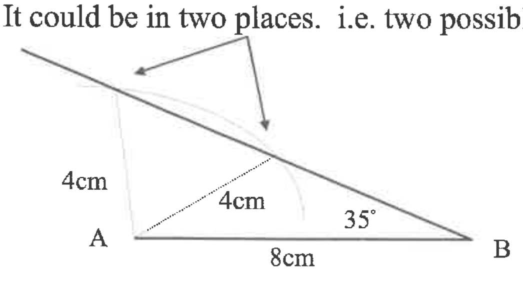
When given two sides and an angle which is not included beware of the 'ambiguous case' when trying to find an angle.

E.g. For triangle ABC, AB = 8 cm AC = 4 cm and angle B = 35°

First sketch the side with the given angle at one end. Sketch in the unknown side at the given angle.



Now AC = 4 cm It could be in two places. i.e. two possible triangles.



This only happens if the side not next to the angle is smaller than the other given side and the given angle is acute. Otherwise you will only get one triangle.

105. In triangle FGH, FG = 3, GH = 4 and angle F = 40°
Find the angles at G and H.

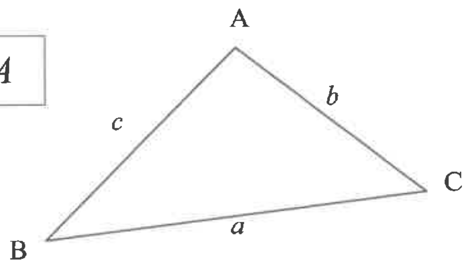
Sketch FG and the angle at F and label them.
GH > FG therefore only one solution.
Complete the sketch.

Use the sine rule to work out angle H and then angles in a triangle add to give 180° to find G.

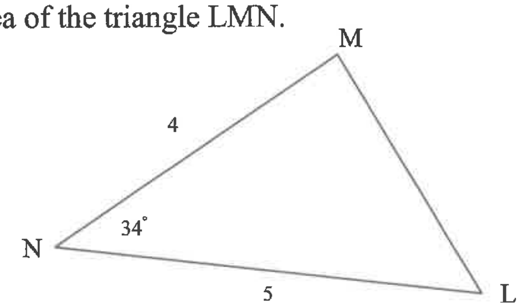
The cosine rule:

Easiest form to use to find a side $a^2 = b^2 + c^2 - 2bc\cos A$

Easiest form to use to find an angle $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$



106. (a) Find LM for the given triangle.
(b) Workout the area of the triangle LMN.



Cannot use the sine rule as you will have two unknowns in any equation.

(a) $LM^2 = LN^2 + NM^2 - 2 \times LN \times NM \times \cos N$

Substitute in values.

$=$

$=$

Careful! Work out
$2 \times LN \times NM \times \cos N$
before subtracting from the rest.

LM =

(b) Area of triangle =

Use Area = $\frac{1}{2} a\, b \sin C$

107. A triangle has sides 4 cm , 5 cm and 6 cm. Find the angle opposite the 6 cm side.

Sketch the triangle.

Use the formula. Substitute known values. Careful the $-a^2$ *is the side opposite the angle A.*

If cos A is negative it just means the angle is greater than 90°.

OR

You can use the other formula – substitute in and rearrange.

Sine rule and cosine rule questions.

108. In triangle ABC, AB = 5.5 cm, AC = 2.7 cm and the angle at B is 20°
Find the two possible values of the angle at C.

109. Two towers P and Q are 600 m apart with P being due East of Q. A church, C, is on a bearing of 210° from P and 125° from Q. Find the distance of the church from P and Q.

110. In a survey P,Q and R are three points on a plot of land. If PQ = 200 m, QR = 172 m and the angle PQR = 68°, find PR and the angles RPQ and QRP.

111. The points A, B and P lie in a straight line on level ground. A pagoda at P is such that the angles of elevation of the top of the pagoda from A and B are 30° and 40° respectively. If AB = 25 m, calculate the height of the pagoda.

Mixed questions 2

112. Solve $2\sin x = 1$ for $-360^\circ < x < 360^\circ$

113. Solve the following equation for $-180^\circ \leq x \leq 180^\circ$
$10\cos 2x = 2$

114. Solve $\sin x = \cos x$ for $0^\circ \leq x \leq 360^\circ$

115. Solve $2\cos^2 x - 1 = 0$ for $0^\circ \leq x \leq 360^\circ$

116. Solve the equation $1 - 2\sin^2\theta + \cos\theta = 0$ for values of θ between 0° and 360°.

117. Work out the value of $\cos 30^\circ + \sin 45^\circ$ giving your answer as a single fraction, in surd, form with a rational denominator.

118. Expand $(2x - 1)^5$

119. Expand $(1-3x)^7$ as far as the term in x^3 Obtain an approximation for 0.997^7 to 6 decimal places by using a suitable value of x in this expansion. (Show your working.)

120. Solve $\sin x + 3\cos x = 0$ for $0^{\circ} \leq x \leq 360^{\circ}$ Give your answers to one decimal place.

121. A triangle, ABC, has length AB = 25 m. Given that angle BAC = 55° and angle ABC = 75°, calculate the area of the triangle to 3 significant figures.

122. The points P, Q and R have coordinates (1, 10), (2, 5) and (12, 3) respectively. Find the angle PRQ to one decimal place.

123. Solve $5\sin x + 3 = 0$ for $360^{\circ} \leq x \leq 720^{\circ}$ Give your answers to one decimal place.

124. $\cos\theta = \frac{7}{25}$ and $180^{\circ} < \theta < 360^{\circ}$ Find the value of $\sin\theta$ and $\tan\theta$ in the form $\frac{a}{b}$, where a and b are integers.

DIFFERENTIATION

Reminder:

$$y = \mathrm{k}x^n \Rightarrow \frac{dy}{dx} = \mathrm{k}\,n\,x^{n-1}$$

Where k is a constant (i.e. 2 or 3 or −2 or − 1 or ...... or ½ or... etc and n is rational).

$$y = \mathrm{c} \Rightarrow \frac{dy}{dx} = 0$$

Where c is a constant (i.e. 2 or 3 or − 2 or − 1 or ...... or ½ or... etc).

$$y = \mathrm{f}(x) + \mathrm{g}(x) \Rightarrow \frac{dy}{dx} = \mathrm{f}'(x) + \mathrm{g}'(x)$$

Differentiate each term.

125. Find $\frac{dy}{dx}$ in each of the following:

(a) $y = x^3 - 5x + 4$

(b) $y = x(x+4)(x-5)$

(c) $y = 2\sqrt{x^3}$

(d) $y = \frac{3}{x^2} + \frac{7}{2\sqrt{x}}$

(e) $y = \frac{2x^2 - 5x}{\sqrt{x}}$

Maximum and minimum points and points of inflection:

Stationary points are where $\frac{dy}{dx} = 0$

Local maximum Local minimum Points of inflection

Turning points are where $\frac{dy}{dx} = 0$ **and** the gradient changes from positive to negative, or negative to positive i.e. Local maximum and local minimum.

126. Find the stationary points for the curve $y = x^3 - 6x^2 + 9x - 1$ and distinguish between them. Without finding where the curve cuts the x axis, sketch the curve.

Complete: $\frac{dy}{dx} =$

$0 =$

$x =$ or

Find corresponding y coordinates. $x =$, $y =$ and $x =$, $y =$

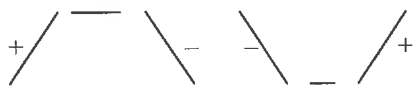
Differentiate.

Put equal to zero.

Solve to find the values of x.

Determine whether a local maximum or a local minimum for each of the values of x.

Find $\frac{dy}{dx}$ just before and just after your x value. Then use the following



max *min*

Sketch the curve. Mark in the points where the curve cuts the y axis and the stationary points just found.
Label these points.

The second derivative:

This is found by differentiating $\frac{dy}{dx}$ It is written $\frac{d^2y}{dx^2}$ or $f''(x)$ if using the function notation.

It can be used to determine the nature of stationary points (i.e. where $\frac{dy}{dx} = 0$)

For a maximum $\frac{d^2y}{dx^2}$ is negative. For a minimum $\frac{d^2y}{dx^2}$ is positive.

If $\frac{d^2y}{dx^2} = 0$ then look at the gradient just before and just after the point.

For max / — \ gradient changes from +ve to –ve as x increases - so gradient graph goes down \ as x increases. ∴ The gradient of the gradient graph is negative.
i.e. $\frac{d^2y}{dx^2}$ is negative for a maximum.
Similarly $\frac{d^2y}{dx^2}$ is positive for a minimum.

127. Find $\frac{d^2y}{dx^2}$ if $y = 2x^4 + 5x^2$	
Complete: $\frac{dy}{dx} =$	*Differentiate.*
$\frac{d^2y}{dx^2} =$	*Differentiate $\frac{dy}{dx}$*
128. If $y = 5x^2 - 2x$ show that $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 20x + 6$	
Complete: $\frac{dy}{dx} =$	*Differentiate.*
$\frac{d^2y}{dx^2} =$	*Differentiate $\frac{dy}{dx}$*
$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} =$	*Add $\frac{d^2y}{dx^2}$ and 2 lots of $\frac{dy}{dx}$*
$=$	*Tidy up.*
129. Find the stationary points on the curve $y = 2 + 3x^2 - x^3$ and distinguish between them.	
Complete: $\frac{dy}{dx} =$	*Differentiate.*
For stationary points $\frac{dy}{dx} = 0$	
$= 0$	
$(\qquad) = 0$	*Factorise - take 3x out as a factor.*
$x =$ or	*Find x.*
$\frac{d^2y}{dx^2} =$	*Differentiate again.*
At $x =$ $\frac{d^2y}{dx^2} =$ i.e ve	
∴ (,)	*Max or min? Give the actual point. (i.e. work out the y coordinate as well.)*
At x = $\frac{d^2y}{dx^2} =$ i.e ve	
∴ (,)	*Max or min? Give the actual point. (i.e. work out the y coordinate as well.)*

Increasing and decreasing functions:

For increasing functions $\dfrac{dy}{dx} > 0$ For decreasing functions $\dfrac{dy}{dx} < 0$

130. Show that $f(x) = x^2 - 5x + 6$ is an increasing function for values of $x > 2.5$	
Complete: $f'(x) =$	*Differentiate.*
$0 =$ $\Rightarrow$ $x =$	*Find x when $f'(x) = 0$*
$f''(x) =$ therefore a	*Find the second differential. Max or Min? > 0 or < 0 ?*
for $x < 2.5$ $f'(x)$	
for $x = 2.5$ $f'(x)$	
for $x > 2.5$ $f'(x)$	*> 0 or < 0 ?*
therefore $f(x)$ is an increasing function for $x > 2.5$	

Differentiation questions:

131. Find the coordinates of the stationary points on the curve $y = 3x - x^3$ and distinguish between them. Use this information to sketch the curve.

132. Given that $y = x^3 + 3x^2 - 24x + 5$
   (a) Find the gradient function.
   (b) Find the values of x for which the gradient is zero.
   (c) Show that one of these points is a minimum and the other is a maximum.
   (d) Find the corresponding y values and sketch the graph.

133. Find the stationary points of the curve $y = 2x^4 - 4x^3 + 5$ and state their nature.

134. A square thin sheet of metal has sides of length 10 cm. Equal square pieces are cut from each corner and the remaining piece is bent into an open box. Find the maximum volume of the box.

135. The length of a rectangular block of ice is twice its width. If the volume is 144 cm^3 find the minimum surface area of the block, to three significant figures.

INTEGRATION

Reminder:

$\int ax^n dx = \frac{ax^{n+1}}{n+1} + c \quad n \neq -1$

Example: $\int 2x^3 dx = \frac{2x^4}{4} + c = \frac{x^4}{2} +$

$\int k\, dx = kx + c$

Example: $\int 3\, dx = 3x + c$

$\int f(x) + g(x)\, dx = \int f(x)\, dx + \int g(x)\, dx$

Example:
$\int 2x^2 + 3x\, dx = \frac{2x^3}{3} + \frac{3x^2}{2} + c$

Work out the following:

136. (a) $\int 8x^3\, dx$ (b) $\int 3x - 2\, dx$ (c) $\int x + \frac{2}{x^2} - \sqrt{x}\, dx$

Definite integrals:

Example: Evaluate $\int_1^2 5x^2\, dx = \left[\frac{5x^3}{3}\right]_1^2$ $= \left[\frac{5.2^3}{3}\right] - \left[\frac{5.1^3}{3}\right]$ $= \frac{40}{3} - \frac{5}{3} = \frac{35}{3}$	*Integrate. No need for the constant.* *Substitute in the limits and subtract.* *Tidy up and work out.*
137. Evaluate $\int_2^5 7\, dx =$	*Integrate. No need for the c.* *Substitute in the limits and subtract.* *Tidy up and work out.* *Here you could sketch the curve and work out the area from your sketch – a rectangle!*
138. Evaluate $\int_1^3 4x\, dx =$	*Integrate. No need for the c.* *Substitute in limits and subtract.* *Work out.*
139. Evaluate $\int_0^2 x^2 + 5\, dx =$	*Integrate. No need for the c.* *Substitute in limits and subtract.* *Work out.*
140. Evaluate $\int_1^4 4\sqrt{x} - \frac{2}{x^2}\, dx =$	*First write each term as powers of x.* *Integrate each term.* *This is a definite integral - no constant of integration required.* *Tidy up.* *Substitute in the limits.* *Work out.*

Area under a curve:

The area between a curve $y = \mathrm{f}(x)$ and the x axis for $x = a$ to $x = b$ is given by:

Area between curve and x axis $= \int_a^b y\,dx = \int_a^b f(x)\,dx$

use the function of x in the integral.

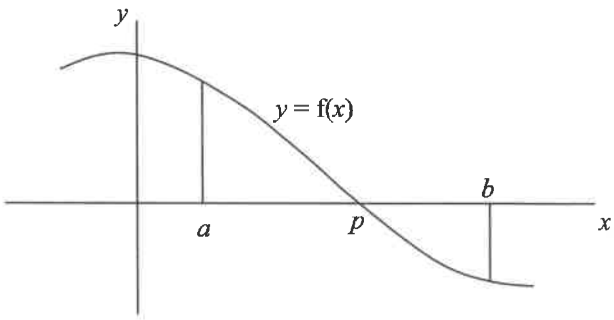
Areas below the x axis are negative.

Sketch the curve if possible.

For area between the curve $y = \mathrm{f}(x)$ and the x axis for $x = a$ to $x = b$ as shown below you will need to work out:

$\int_a^p f(x)\,dx$ and $\int_p^b f(x)\,dx$

Make both areas positive and add them together.



You could be asked to work out the integral $\int_a^b f(x)\,dx$ If so, then do not worry about signs, just work it out!

141. Find the area between the curve $y = 9 - x^2$ and the x axis. Complete: $0 = 9 - x^2$	*Find intersections with the x axis i.e. where y = 0*
$=$ $x =$ and $x =$	*Sketch the curve.*
Area $= \int \qquad dx$	*Put in the function and the limits.*
$= \Big[\qquad \Big]$	*Integrate and write in limits.*
$= [\qquad] - [\qquad]$	*Substitute in the limits.*
$=$	*Work out the answer.*
142. Find the area between the curve $y = x(x-1)$ and the x axis from $x = 0$ to $x = 2$	*Find intersections with the x axis i.e. where y = 0* *Sketch the curve.* *Note: partly above and partly below the x axis. Work out the area above the x axis and the area below the x axis separately, make them both positive and add them.*

143. $f(x) = x^2 - 4x + 7$

(a) Complete the square on $f(x)$ and hence sketch the curve $y = f(x)$

(b) Find the area between the curve $y = f(x)$, the coordinate axes and the line $x = 3$

Area between a straight line and a curve:

Sketch the curve and line, work out the points of intersection. From your sketch you should be able to find the required area.

144. Find the area of the region enclosed by $y = x^2 + 2$ and $y = 11$

Area between line and curve =

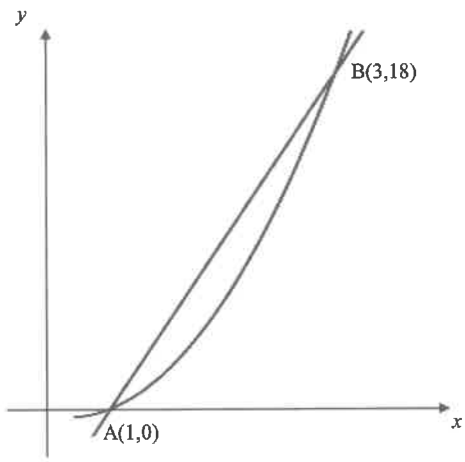
Sketch the line and curve.

Work out the points of intersection.

With respect to the x axis:
Find area under the curve between the points of intersection by integration.
Find area under the line between the points of intersection.

Just subtract. Careful with the minus.

145. The curve with equation $y = x^2 + 5x - 6$ is sketched below.
The curve cuts the x-axis at the point $A(1, 0)$ and the point $B(3, 18)$ lies on the curve.
Find the area between the curve and the line AB.



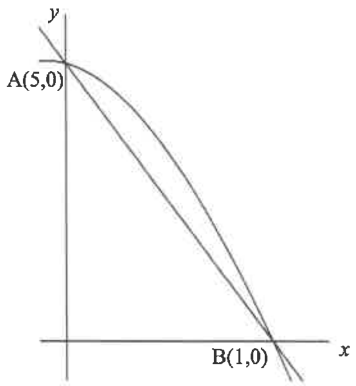
Integration questions.

146. Evaluate

(a) $\int_{-1}^{1} x^2(x-6)\ dx$ (b) $\int_{-1}^{1} 1-t^3 dt$

147. Evaluate $\int_{1}^{3} \frac{3x^2+4x}{2x}\ dx$

148. The curve with equation $y=x^3-5x^2-x+5$ is sketched below. The curve cuts the y axis at the point $A\,(5,0)$ and the x axis at $B\,(1,0)$. Find the area between the curve and the straight line AB to 3 significant figures.



149. Sketch the curve $y=(x+2)(x-1)(x-4)$

(a) Find the area between the curve and the x axis for $x=-2$ to $x=1$

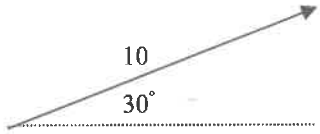
(b) Find the area between the curve and the x axis for $x=1$ to $x=4$

VECTORS

Quantities that have **magnitude** (size) and **direction** are called **vectors**. e.g. displacement, velocity, acceleration, force,…
They can be represented by a line segment of length equal to the magnitude and direction equal to the direction of the vector.

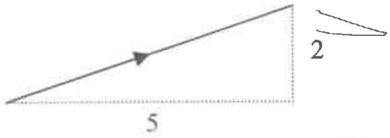
Magnitude direction form of a vector:

This vector has magnitude 10
and direction $+30^\circ$ from the positive x direction
(anti-clockwise from the positive x direction).
(The direction could be given as a bearing - in this case 060°)
[It could represent a force of 10 Newtons at $+30^\circ$
or a velocity of 10 ms^{-1} at $+30^\circ$, etc.]



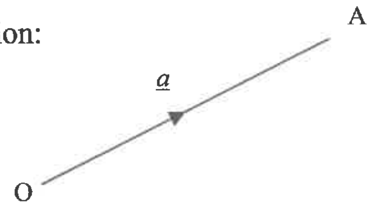
Component form of a vector:

This vector is written $\begin{pmatrix}5\\2\end{pmatrix}$ or $5\mathbf{i}+2\mathbf{j}$



Where **i** is a unit vector (a vector of magnitude 1) in the x direction and **j** is a unit vector in the y direction.

Notation:



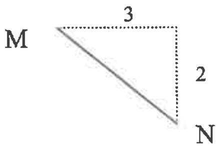
in bold type

Vector a is written $\underset{\sim}{a}$ or $\overrightarrow{OA}$ or $\boldsymbol{a}$ in books and exam papers.

The magnitude (or modulus) of this vector is written a or $|\underset{\sim}{a}|$ or $|\overrightarrow{OA}|$

not bold

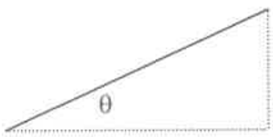
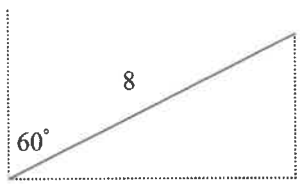
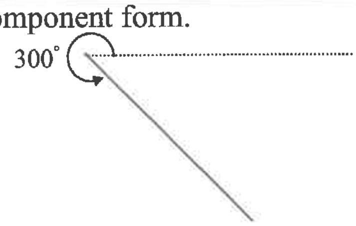
and



$$\overrightarrow{MN}=\begin{pmatrix}3\\-2\end{pmatrix}=3\mathbf{i}-2\mathbf{j}$$

$$\overrightarrow{NM}=\begin{pmatrix}-3\\2\end{pmatrix}=-3\mathbf{i}+2\mathbf{j}$$

Complete:

150. Find the magnitude and direction of the vector $5\mathbf{i}+2\mathbf{j}$ Magnitude = $\lvert 5\mathbf{i}+2\mathbf{j}\rvert$ = $\tan\theta =$ $\theta =$ $\therefore$ direction is 	*Always sketch the vector. Especially any with negatives in - as it is easy to get the direction wrong. Write in the lengths 5 and 2.* *Use Pythagoras to obtain the magnitude.* *Use tan to obtain an angle and hence the direction. You must always state the direction clearly - angle that it makes with the positive x direction, or bearing, or….*
151. Write the vector with magnitude 8 and direction given by the bearing 060° in component form. 	*Again sketch the vector.* *Draw in the right angled triangle as shown and then use trig to work out the lengths of the horizontal and vertical lines. Then write in* ***i j*** *form.* ***i*** *and* ***j*** *will both be positive.*
152. Write the vector $(20, 300^\circ)$ in component form. 	*Here the vector is given in polar form i.e. magnitude, direction [angle from +ve x direction]* *Sketch the vector.* *Draw in the right angled triangle and the use trig to work out the lengths of the horizontal and vertical lines. Then write in* ***i j*** *form.* *You can see from the triangle that* ***i*** *part is +ve and the* ***j*** *part is –ve.*

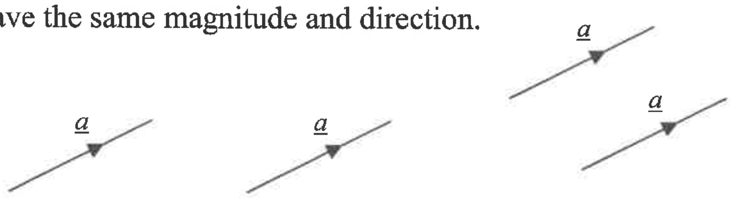
153. Find the magnitude and direction of the vector $-2\mathbf{i}-3\mathbf{j}$

Sketch the vector - as it is easy to get the direction wrong. Write in the lengths 2 and 3.
Use Pythagoras to obtain the magnitude.

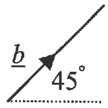
Use tan to obtain an angle and hence the direction. You must always state the direction clearly - angle that it makes with the positive x direction, or bearing, or....

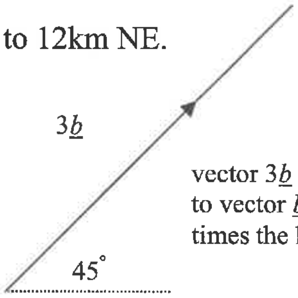
Equal vectors:

Equal vectors have the same magnitude and direction.



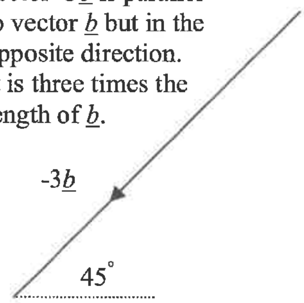
Multiplying by a scalar:
'walk 4km NE' three times is equivalent to 12km NE.





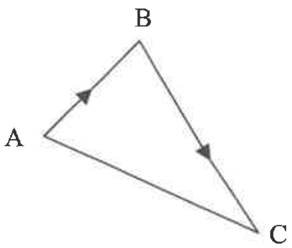
vector $3\underline{b}$ is parallel to vector $\underline{b}$ and three times the length.

vector $-3\underline{b}$ is parallel to vector $\underline{b}$ but in the opposite direction. It is three times the length of $\underline{b}$.



Adding vectors:

$\overrightarrow{AB}=\begin{pmatrix}2\\2\end{pmatrix}$ $\overrightarrow{BC}=\begin{pmatrix}3\\-4\end{pmatrix}$



$$\overrightarrow{AB}+\overrightarrow{BC}=\begin{pmatrix}2\\2\end{pmatrix}+\begin{pmatrix}3\\-4\end{pmatrix}=\begin{pmatrix}5\\-2\end{pmatrix}$$

$$\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$$

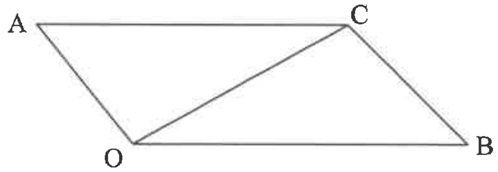
'followed by' 'is equivalent to'

154. For the given diagram complete the following:

$\overrightarrow{OC}=\overrightarrow{OA}+$

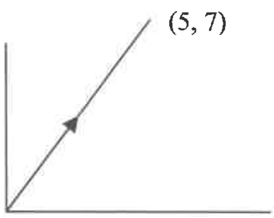
$\overrightarrow{OC}=\overrightarrow{OB}+$

$\overrightarrow{CB}=\overrightarrow{CO}+$ $\qquad\overrightarrow{BC}=$



Position vectors:
These are vectors which start at the origin.

(5, 7) has position vector $5\mathbf{i}+7\mathbf{j}$



155. If $A(2,4)$ and $B(7,1)$ find $\overrightarrow{AB}$ and $\overrightarrow{BA}$
Complete:

Sketch the points A and B and the lines OA, OB and AB.

$\overrightarrow{AB}=\overrightarrow{AO}+\overrightarrow{OB}$ $\qquad\overrightarrow{BA}=\qquad+$

$=\begin{pmatrix}\ \\ \ \end{pmatrix}+\begin{pmatrix}\ \\ \ \end{pmatrix}=\begin{pmatrix}\ \\ \ \end{pmatrix}$ $\qquad=\begin{pmatrix}\ \\ \ \end{pmatrix}+\begin{pmatrix}\ \\ \ \end{pmatrix}=\begin{pmatrix}\ \\ \ \end{pmatrix}$

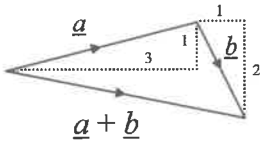
Subtracting vectors:

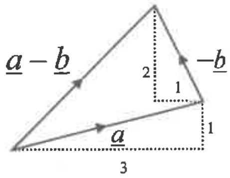
When drawing one vector subtracted from another vector you *add the negative of the vector* to be subtracted.

$\underline{a} = 3\mathbf{i} + \mathbf{j}$ and $\underline{b} = \mathbf{i} - 2\mathbf{j}$

$\underline{a} + \underline{b} = 3\mathbf{i} + \mathbf{j} + \mathbf{i} - 2\mathbf{j} = 4\mathbf{i} - \mathbf{j}$

$\underline{a} - \underline{b} = 3\mathbf{i} + \mathbf{j} - (\mathbf{i} - 2\mathbf{j}) = 2\mathbf{i} + 3\mathbf{j}$





$\underline{a} + (-\underline{b})$ i.e. $\underline{a}$ followed by $-\underline{b}$

Unit vectors:

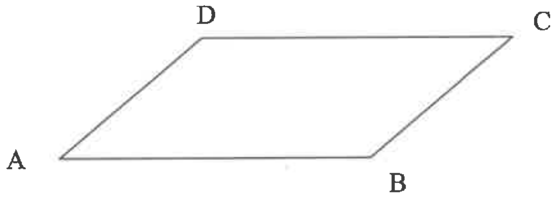
To find a unit vector in a given direction, divide the vector by its magnitude.

If $\underline{r} = 3\mathbf{i} + 4\mathbf{j}$, then a unit vector in the same direction is $\hat{\mathbf{r}} = \dfrac{\mathbf{r}}{|\mathbf{r}|} = \dfrac{3\mathbf{i}+4\mathbf{j}}{\sqrt{9+16}} = \frac{1}{5}(3\mathbf{i}+4\mathbf{j}) = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$

Complete:

156. ABCD is a parallelogram with $\overrightarrow{AB} = \underline{p}$ and $\overrightarrow{AD} = \underline{q}$

Find the following vectors in terms of $\underline{p}$ and $\underline{q}$



$\overrightarrow{DC} =$ $\qquad$ $\overrightarrow{BC} =$

$\overrightarrow{BD}$

If M is the mid point of BC and N the mid point of DC, find

$\overrightarrow{BM}$

$\overrightarrow{AM}$

$\overrightarrow{AN}$

$\overrightarrow{MN}$

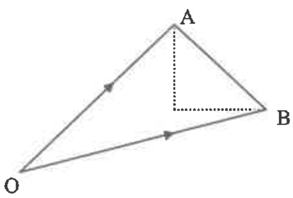
157. Find a unit vector in the direction of $5\mathbf{i} - 12\mathbf{j}$

Finding the distance between two points using vectors:

Find the vector between the two points and then work out the magnitude of this vector.

Example: The position vectors of two points A and B are $6\mathbf{i} + 8\mathbf{j}$ and $10\mathbf{i} + 5\mathbf{j}$ Find the distance between the two points.

$$\overrightarrow{AB} = -(6\mathbf{i} + 8\mathbf{j}) + (10\mathbf{i} + 5\mathbf{j})$$
$$= (10 - 6)\mathbf{i} + (5 - 8)\mathbf{j}$$
$$= 4\mathbf{i} - 3\mathbf{j}$$
$$|\overrightarrow{AB}| = \sqrt{4^2 + (-3)^2} = 5$$

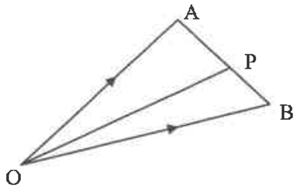


Always draw a sketch so that you can 'see' what you are doing.

158. The position vectors of two points A and B are $-8\mathbf{i} + 3\mathbf{j}$ and $-3\mathbf{i} - 5\mathbf{j}$ Find the distance between the two points to three significant figures.

Finding the point dividing a line in a given ratio:

Example: The position vectors of two points A and B are $6\mathbf{i} + 8\mathbf{j}$ and $10\mathbf{i} + 5\mathbf{j}$
If P divides AB in the ratio 2:1 find P.



$$\overrightarrow{AB} = -(6\mathbf{i} + 8\mathbf{j}) + (10\mathbf{i} + 5\mathbf{j})$$
$$= 4\mathbf{i} - 3\mathbf{j}$$
$$AP : PB = 2:1 \quad \therefore \overrightarrow{AP} = \tfrac{2}{3}\,\overrightarrow{AB}$$
$$\overrightarrow{AP} = \tfrac{2}{3}(4\mathbf{i} - 3\mathbf{j}) = \tfrac{8}{3}\mathbf{i} - 2\mathbf{j}$$
$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} = 6\mathbf{i} + 8\mathbf{j} + \tfrac{8}{3}\mathbf{i} - 2\mathbf{j} = \tfrac{26}{3}\mathbf{i} + 6\mathbf{j}$$

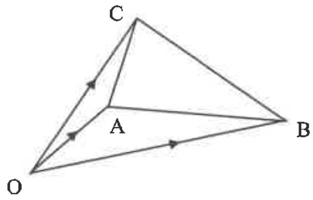
So point P $(\frac{26}{3}, 6)$

Finding the angle between two vectors:

Find the magnitude of each vector and then use the cosine rule to find the required angle.

Example: The position vectors of three points A, B and C are $4\mathbf{i} + 4\mathbf{j}$, $9\mathbf{i} + 3\mathbf{j}$ and $5\mathbf{i} + 7\mathbf{j}$. Find the angle ACB.

$$|\overrightarrow{AB}| = |-(4\mathbf{i} + 4\mathbf{j}) + 9\mathbf{i} + 3\mathbf{j}| = |5\mathbf{i} - \mathbf{j}| = \sqrt{5^2 + 1^2} = \sqrt{26}$$
$$|\overrightarrow{AC}| = |-(4\mathbf{i} + 4\mathbf{j}) + 5\mathbf{i} + 7\mathbf{j}| = |\mathbf{i} + 3\mathbf{j}| = \sqrt{1^2 + 3^2} = \sqrt{10}$$
$$|\overrightarrow{BC}| = |-(9\mathbf{i} + 3\mathbf{j}) + 5\mathbf{i} + 7\mathbf{j}| = |-4\mathbf{i} + 4\mathbf{j}| = \sqrt{4^2 + 4^2} = \sqrt{32}$$
$$\cos(\angle ACB) = \frac{10 + 32 - 26}{2\sqrt{10}\sqrt{32}} = 0.44721$$
$$\angle ACB = 63.4^\circ$$



Always draw a sketch so that you can 'see' what you are doing.

159. The position vectors of three points P, Q and R are $-8\mathbf{i} - 3\mathbf{j}$, $6\mathbf{i} - 5\mathbf{j}$ and $-\mathbf{i} + 3\mathbf{j}$ Find the angle PRQ to one decimal place.

PROOF

Proof by deduction:

Consecutive numbers can be written $n, n+1, n+2, n+3$, etc
Multiples of 2 can be written $2n$, multiples of 3 can be written $3n$, etc
Even numbers can be written $2n$
Odd numbers can be written $2n - 1$

These are true for positive integer values of n. $n = 1, 2, 3$, etc

Prove: even number × even number = even number Proof: even numbers can be written 2n or 2m or 2p etc, where n, m etc are positive integers. even number × even number $= 2n \times 2m$ $= 4nm$ $= 2 \times (2mn)$ which is even.	*Start with 2n and 2m. This is so you have any two different even numbers.* *Multiply and aim to get 2 multiplied by something.*
160. Prove: even number × odd number = even number	*Start with 2n and 2m – 1.* *Multiply and aim to get 2 multiplied by something.*

161. Prove: odd number × even number = even number	*Start with $2n-1$ and $2m$. Multiply and aim to get 2 multiplied by something.*
162. Prove: odd number × odd number = odd number	*Start with $2n-1$ and $2m-1$. Multiply and aim to get 2 multiplied by something plus 1.*

Note: The above results can be used in further proofs.

163. p is an odd number, explain why p^2+p will always be even.

164. Prove algebraically that the difference between the cube and the square of any odd integer is even.	*Prove algebraically means you must use algebra.* *Use $2n-1$. Cube and square and subtract. Easiest if you can take out a factor without multiplying everything out.*
165. Here are the first five terms of an arithmetic sequence. 2 7 12 17 22 Prove that the difference between any two terms of the sequence is always a multiple of 5.	*Work out how to generate the n^{th} term.* *As **any** two terms - you need the difference between the n^{th} term and the m^{th} term.*
166. Prove algebraically that $(2n-1)^2-(2n-1)$ is an even number for all positive integer values of n.	*Prove algebraically means you must use algebra.* *Multiply out and collect like terms or take $(2n-1)$ out as a factor.*

167. Using completion of the square, prove that n^2+4n+5 is positive for all values of n.

168. Prove that $xy \le \frac{x^2+y^2}{2}$ for all real x.

Proof by exhaustion:

This involves trying all the options to show a statement is true.

Example: Prove that if n is an integer and $3 \le n \le 6$ then n^2+1 is not divisible by 4.

$n=3$	$n^2+1=9+1=10$	not divisible by 4	
$n=4$	$n^2+1=16+1=17$	not divisible by 4	
$n=5$	$n^2+1=25+1=26$	not divisible by 4	
$n=6$	$n^2+1=36+1=37$	not divisible by 4	Proved

169. Prove that $n^2 \ge 2^n$ for $2 \le n \le 4$, where n is an integer.

Disproof by counter example:

This involves finding a value/s that disproves the statement.

Example: Prove that the statement ' n^2-n+5 is a prime number for all values of n for $n \ge 0$ ' is not true.

Search for a value:
n = 1 $n^2-n+5=1-1+5=5$ true
n = 2 $n^2-n+5=4-2+5=7$ true
n = 3 $n^2-n+5=9-3+5=11$ true
n = 4 $n^2-n+5=16-4+5=17$ true
n = 5 $n^2-n+5=25-5+5=25$ not prime as $5\times5=25$ $\therefore$ not true.

$\therefore$ 'n^2-n+5 is a prime number for all values of n' is not true.

170. Prove that the statement ' 3^n+2 is a prime number for all values of n for $n \ge 0$' is not true.

171. Disprove the statement 'the sum of n consecutive integers is divisible by n, where n is a positive integer'.

172. "For any positive whole number, n, the value of $4n^2+7$ is a prime number." Find a value for n that disproves this statement.

Mixed Questions 3

173. The position vectors of three points A, B and C are $2\mathbf{i} + \mathbf{j}$, $-3\mathbf{i} - 3\mathbf{j}$ and $-2\mathbf{i} + 4\mathbf{j}$ respectively. Find the vectors $\overrightarrow{BA}$, $\overrightarrow{BC}$ and $\overrightarrow{AC}$. Hence find the angle ABC to one decimal place

174. $y = x^2 + \dfrac{54}{x}$ Find the coordinates of any turning points on the curve.

175. Given that the point A has position vector $\mathbf{i} + 4\mathbf{j}$ and the point B has position vector $5\mathbf{i} - 2\mathbf{j}$
(a) Find the vector AB
(b) Find $|\overrightarrow{\text{AB}}|$ Give your answer as a simplified surd.

176. Prove that the sum of any four consecutive positive integers cannot be divided by 4.

177. Find $|\boldsymbol{a} - \boldsymbol{b}|$ where $\boldsymbol{a} = 2\mathbf{i} - 3\mathbf{j}$ and $\boldsymbol{b} = \mathbf{i} - 5\mathbf{j}$

178. The position vectors of two points A and B are $\mathbf{i} - 3\mathbf{j}$ and $-2\mathbf{i} + 5\mathbf{j}$ relative to the origin O. Find the angle between the two vectors, AOB, to one decimal place.

179. Evaluate $\displaystyle\int_1^{\sqrt{2}} \frac{3x^4 + 4}{2x^3}\, dx$

Mixed Questions 4

180. Find the coefficient of x^3 in the binomial expansion of $(2x+3)^6$

181. Find the centre and radius of the circle $x^2+y^2+8x-6y+21=0$

182. Solve the equation $2\cos\theta=\sqrt{3}$ for $0^\circ<\theta<360^\circ$

183. Find $\int_0^4 \frac{1}{2}\sqrt{x^3}-\frac{2}{\sqrt{x}}\,dx$

184. Solve the equation $1-2\sin^2\theta-\cos\theta=0$ for $0^\circ\leq\theta\leq 360^\circ$

185. Find the area between the curve $y=x^2$ and the line $y=2x$

186. $f(x)=2x^3-9x^2+14x-8$
   (a) Use the factor theorem to show that $(x-2)$ is a factor of $f(x)$
   (b) Hence show that 2 is the only real root of the equation $f(x)=0$

187. Expand $(1-2x)^7$ in ascending powers of x up to the term in x^3

188. A boat at point P observes two lighthouses A and B at bearings 040° and 330° respectively. If A and B are 8 nautical miles apart and the bearing of B from A is 265°, calculate the size of angle ABP and the distance of the ship from A.

189. Find the enclosed area between the curve $y = 3x^2 - x^3$ and the positive x axis.

190. Show that $(x-3)$ is a factor of $x^3 - 19x + 30$ and find the other linear factors. Solve the equation $x^3 - 19x + 30 = 0$

191. $P = P_0 e^{kt}$ Given that initially $P = 1000$ and $P = 2000$ when $t = 2$, find t when $P = 5000$

192. Solve $\cos 3x = 0$ for $0^{\circ} \leq x \leq 180^{\circ}$

193. A circle passes through A(2, 3) and B(10, 9) where AB is a diameter of the circle. Find the equation of the circle.

194. Evaluate $\int_0^2 x^3 - 5x^2 + 1 \, dx$

195. A right circular cylinder is to be made so that the sum of the radius and height is 24 cm. Show that the maximum volume of the cylinder is $2048\pi \text{ cm}^3$.

196. Solve (a) $5^x = 43$ (b) $e^{x-1} = 50$ (c) $2\ln(4x + 1) = 9$

197. Given that the position vectors of A and B are $9\mathbf{i} - 3\mathbf{j}$ and $-\mathbf{i} + 2\mathbf{j}$

If P divides AB in the ratio 2:3 find the position vector of P.

198. The equation of a curve is $y = x^3 - 6x^2$

(a) Find the coordinates of the stationary points on the curve and determine their nature.

(b) Factorise $x^3 - 6x^2$

(c) Using (a) and (b) sketch the curve.

(d) State the set of values of x for which $x^3 - 6x^2$ is a decreasing function.

199. The variables s and t are believed to satisfy a relationship of the form $s = ka^{-t}$ where k and a are constants. Show that the experimental values given in the table do satisfy the relationship using the graph. Find approximate values for k and a.

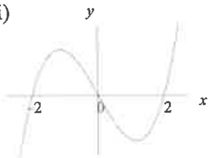
t	1	2	3	4
s	1.75	0.44	0.11	0.03

Answers

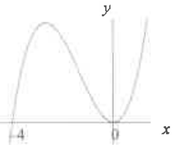
Page3

1. (i) $x^3 - 4x = x(x^2 - 4) = x(x+2)(x-2)$

(ii)

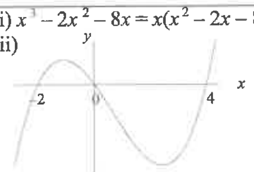


2. $x^3 + 4x^2 = x^2(x+4)$



3. (i) $x^3 - 2x^2 - 8x = x(x^2 - 2x - 8) = x(x-4)(x+2)$

(ii)



4. $x^3 + 2x^2 - 5x = x(x^2 + 2x - 5)$

5. $4x^3 - 7x^2 - 2x = x(4x^2 - 7x - 2) = x(4x+1)(x-2)$

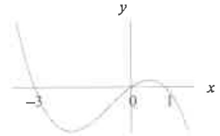
$x(4x+1)(x-2) = 0$

$x = 0, 2$ or $-\frac{1}{4}$

6. (i) $-x^3 - 2x^2 + 3x = -x(x^2 + 2x - 3)$

$= -x(x-1)(x+3)$

(ii)



Page4

7.

$$\begin{array}{r} 2x^2+5x+2 \\ x-5\,\overline{)\,2x^3-5x^2-23x-10} \\ -(2x^3-10x^2) \\ 5x^2 \\ -(5x^2-25x) \\ 2x \\ -(2x-10) \\ 0 \end{array}$$

8.

$$\begin{array}{r} x^2-2x-24 \\ x-1\,\overline{)\,x^3-3x^2-22x+12} \\ -(x^3-x^2) \\ -2x^2 \\ -(-2x^2+2x) \\ -24x \\ -(-24x+24) \\ -12 \end{array}$$

Quotient is $x^2 - 2x - 24$ Remainder is -12

Page5

9. Complete: $f(2) = 8 + 4k - 10 - 1 = 0$ $k = 3/4$

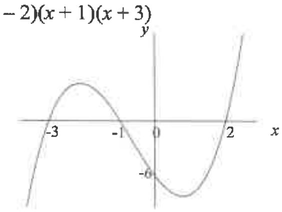
10. Complete: $f(1) = -8$

$f(2) = 0$

$f(-1) = 0$

$f(-3) = 0$

$(x-2)(x+1)(x+3)$



11. Complete: $f(1) = 1$ $f(2) = 0$

$(x-2)(x^2+x-3)$

$(x-2)(x^2+x-3) = 0$

$x = 2$ or $x = \dfrac{-1 \pm \sqrt{1+12}}{2}$

$x = 2$ or $x = \dfrac{-1+\sqrt{13}}{2}$ or $x = \dfrac{-1-\sqrt{13}}{2}$

Page6

12. $f(1) = 0$, $f(-3) = 0$, $f(5) = 0$

$(x-1)(x+3)(x-5) = 0$ $\therefore$ $x = -3, 1, 5$

13. $f(½) = 2\times{}^1/_8 - {}^1/_4 - 2\times{}^1/_2 + 1 = 0$

$\therefore$ $(2x-1)$ is factor

14. $(x-3)^2 + (y-5)^2 = 16$

$x^2 + y^2 - 6x - 10y + 18 = 0$

15. $x^2 + (y-4)^2 = 25$, $y = 0$ $x^2 + 16 = 25$

$x = 3$ or -3

The circle intersects the x axis at (3,0) and (–3,0)

16. $x = 0$, $(y-2)^2 = 25 - 16$ $y = 5$ or -1

The circle intersects the y axis at (0,5) and (0,–1)

Page7

17. Mid point (4, 5)

Length AB $= \sqrt{[(3-5)^2 + (7-3)^2} = \sqrt{20} = 2\sqrt{5}$

Radius $= \sqrt{5}$

$(x-4)^2 + (y-5)^2 = 5$

18. $(x+4)^2 - 16 + (y + {}^3/_2)^2 - {}^9/_4 - 2 = 0$

$(x+4)^2 + (y + {}^3/_2)^2 = {}^{81}/_4$

Centre $(-4, -{}^3/_2)$ radius ${}^9/_2 = 4.5$

19. Midpoint (6,10)

AB gradient $= -4/8 = -½$ perp gradient $= 2$

$\dfrac{y-10}{x-6} = 2$ $y - 10 = 2x - 12$ $y = 2x - 2$

x axis $y = 0$ $2x - 2 = 0$ $x = 1$ C(1,0)

Radius CA$= \sqrt{(1^2 + 12^2)} = \sqrt{145}$

$(x-1)^2 + y^2 = 145$

$x^2 + y^2 - 2x - 144 = 0$

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20. $(x-1)^2 + (8-x-2)^2 = 25$

$(x-1)^2 + (6-x)^2 = 25$

$x^2 - 2x + 1 + 36 - 12x + x^2 - 25 = 0$

$2x^2 - 14x + 12 = 0$

$2(x-1)(x-6) = 0$

$x = 1$ or 6

Intersects at $x = 1$ $y = 7$ and $x = 6$ $y = 2$

21. $x^2 + (x+3)^2 - 8x - 6(x+3) + 17 = 0$

$x^2 + x^2 + 6x + 9 - 8x - 6x - 18 + 17 = 0$

$2x^2 - 8x + 8 = 0$

$2(x-2)(x-2) = 0$

$x = 2$ twice

Only one value of x (two equal roots) therefore the line touches the circle.

(Or work out '$b^2 - 4ac$' for the quadratic equation. If this equals 0 then the line touches the circle.)

22. PQ $= \sqrt{(9-1)^2 + (-4-2)^2} = 10$

Gradient PQ $= \dfrac{-4-2}{9-1} = -\dfrac{3}{4}$

Gradient QR $= \dfrac{12--4}{21-9} = \dfrac{4}{3}$

$-\frac{3}{4} \times \frac{4}{3} = -1$ therefore PQR is a right angle

Equation of circle $(x-1)^2 + (y-2)^2 = 100$

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23. $\log_{10}100 = 2$ since $10^2 = 100$

$\log_{10}1{,}000{,}000 = 6$ since $10^6 = 1{,}000{,}000$

$\log_{10}1 = 0$ since $10^0 = 1$

$\log_{10}0.01 = -2$ since $10^{-2} = 0.01$

24. $\log_3 9 = 2$ since $3^2 = 9$

$\log_{16}4 = ½$ since $16^{\frac{1}{2}} = 16^{0.5} = 4$

Page10

25. $\log_{10}5^n = \log_{10}60$

Using law 3, $n\log_{10}5 = \log_{10}60$

Using a calculator, $n = \log_{10}60/\log_{10}5$

and hence $n = 2.54396$ (to 5 decimal places)

26. $0.5 = ½$ so $\log_x 0.5 = \log_x ½ = \log_x 1 - \log_x 2$

$= -\log_x 2 = -0.6$

27. $2\log 3 = \log 3^2 = \log 9$,

$\frac{1}{2}\log 25 = \log 25^{\frac{1}{2}} = \log 5$

and $\log\frac{1}{3} = -\log 3$

So $\log 9 + \log 5 - \log 4 - \log 3 = \log\dfrac{9\times5}{4\times3}$

$= \log\dfrac{45}{12} = \log\dfrac{15}{4}$

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28. $1.05^n \times £200 = £300$

$1.05^n = \dfrac{300}{200} = 1.5$

$\log 1.05^n = \log 1.5$

$n\log 1.05 = \log 1.5$

$n = \dfrac{\log 1.5}{\log 1.05} = 8.31$ (2 d.p.)

so the money needs to be left for 9 years before the total is greater than £300.

29. $47 = 10^{1.672}$

30. $\log_{10}(14\times5)/7 = \log_{10}10 = 1$

31. $3\log x + \log x = 4\log x$ or $\log x^4$

32. $3\log x / \log x = 3$

33. $\log 3^n = \log 250$

$n = \log 250 \div \log 3 = 5.03$

34. $\log_a x^3 = \log_a 5^3$

$x = 5$

35. $\log y - \log 3 = 2$

$\log {}^y/_3 = 2$

${}^y/_3 = a^2$ $\therefore$ $y = 3a^2$

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36.(a) $\ln 3 = 1.0986$ (b) $\ln 13.5 = 2.6027$

(c) $e^3 = 20.0855$ (d) $e^{1.6} = 4.9530$

37. (a) $\ln\dfrac{x}{2x+1} = \ln x - \ln(2x+1)$

(b) $\ln 8x^2 = \ln 8 + 2\ln x$

(c) $\ln\tan x = \ln\sin x - \ln\cos x$

38. (a) $\ln(2x+1) + \ln A = \ln A(2x+1)$

(b) $1 - \ln x = \ln e - \ln x = \ln(e/x)$

(c) $2\ln x + ½\ln(x+1) = \ln x^2 + \ln(x+1)^{½}$

$= \ln x^2(x+1)^{½}$

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39. (a) $e^x = 5.4$

$\ln e^x = \ln 5.4$

$x = \ln 5.4 = 1.6864 = 1.69$ to 3 s.f.

(b) $e^{2x+1} = 10$

$(2x+1)\ln e = \ln 10$

$2x + 1 = \ln 10$

$x = (\ln 10 - 1)/2 = 0.6513 = 0.651$ to 3 s.f.

(c) $e^{2x} + e^x - 6 = 0$

$y^2 + y - 6 = 0$

$(y+3)(y-2) = 0$

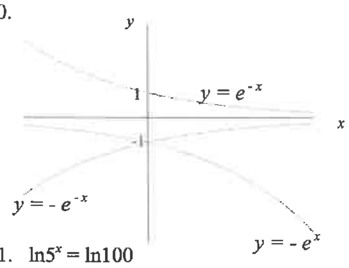
$e^x = -3$ or $e^x = 2$

$\ln e^x = \ln(-3)$ or $\ln e^x = \ln 2$

no solution or $x = \ln 2$

Answer: $x = 0.693$ to 3 s.f.

40.



41. $\ln 5^x = \ln 100$

$x\ln 5 = \ln 100 \Rightarrow x = \ln 100/\ln 5$

$x = 2.86$ (to 3 s.f.)

42. $2x = e^{0.5}$ $x = \dfrac{e^{0.5}}{2} = 0.824$ (to 3 s.f.)

43. (a) $x = \ln 4 = 1.38629 = 1.39$ (to 3 s.f.)

(b) $(x+2)\ln 3 = \ln 86$

$(x+2) = \dfrac{\ln 86}{\ln 3} = 4.05452$

$x = 2.05$ (to 3 s.f.)

44. $500 = 2000\,e^{-0.2T}$

$0.25 = e^{-0.2T}$

$\ln 0.25 = \ln e^{-0.2T}$

$\ln 0.25 = -0.2T\ln e$

$T = \ln 0.25 \div (-0.2) = 6.93$ to 3 sig.fig.

45. $1500 = 1200\,e^{5k}$

$1.25 = e^{5k}$

$\ln 1.25 = 5k\ln e$

$k = \ln 1.25 \div 5 = 0.0446 = 0.045$ to 3 dec.pl.

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46. $1 = 2\,e^{-0.02t}$

$0.5 = e^{-0.02t}$

$\ln 0.5 = -0.02t\ln e$

$t = \ln 0.5 \div (-0.02)$

$= 34.657 = 34.7$ to 3 sig.fig.

47. $15 = Ae^0 \Rightarrow A = 15$

$60 = 15e^k$

$4 = e^k$

$\ln 4 = \ln e^k \Rightarrow k = \ln 4 = 1.39$ to 3 s.f.

$y = 15e^{1.3863t}$ $t = 2$, $y = 15e^{2.7726} = 240$

48. $m = Ae^{-kt}$ $10 = Ae^0 = A$

$9 = 10e^{-4k} \Rightarrow 0.9 = e^{-4k}$

$\ln 0.9 = \ln e^{-4k} = -4k$

$k = 0.02634$

$m = 10e^{-0.02634t}$

$t = 7$, $m = 8.31620 = 8.32$grams to 3 s.f.

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49. (i)

x	1	2	3	5
Y	1.801	2.163	2.525	3.248

(ii) $y = ab^x$

$\log y = \log ab^x$

$\log y = \log a + \log b^x$

$\log y = \log a + x\log b$

$\log y = (\log b)x + \log a$ Plotting $\log y$ against x gives a straight line with gradient $\log b$ and vertical intercept at $\log a$

(iii) $a = 27.5$ (1 d.p.) Accept 25 to 30

$b = 2.3$ (1 d.p.) Accept 2.2 to 2.4

50.

$\log x$	1	2	3	4.5
$\log t$	0.55	0.88	1.22	1.72

$t = kx^n$

$\log t = \log kx^n$

$\log t = \log k + \log x^n$

$\log t = \log k + n\log x$

$\log t = n(\log x) + \log k$ Plotting $\log t$ against $\log x$ gives a straight line with gradient n and vertical intercept at $\log k$

$\log k = 0.2$ $\therefore$ $k = 1.58$ Accept 1.5 to 1.7

Gradient $= (1.72 - 0.55)/3.5 = 1.17/3.5 = 0.33$

$\therefore$ $n = 0.33$ Accept 0.32 to 0.36

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51. $\log 3^x = \log 77$

$x\log 3 = \log 77 \Rightarrow x = \log 77/\log 3$

$x = 3.95$ (to 3 s.f.)

52. $f(1) = 1+4+1-6 = 0$ $\therefore$ $(x-1)$ is a factor

$f(-2) = -8+16-2-6 = 0$ $\therefore$ $(x+2)$ is a factor

$f(-3) = -27+36-3-6 = 0$ $\therefore$ $(x+3)$ is a factor

$x^3 + 4x^2 + x - 6 = (x-1)(x+2)(x+3)$

53. $(x-1)^2 + (y+4)^2 = 25$

$x^2 + y^2 - 2x + 8y - 8 = 0$

$y = 0$, $x^2 - 2x - 8 = 0$

$(x-4)(x+2) = 0$

$x = -2$ or 4

$\therefore$ Cuts x axis at (–2, 0) and (4, 0)

54. $A = 3.4$ $P = 3.4e^{0.05t} = 6.8$

$e^{0.05t} = 2$

$\ln e^{0.05t} = 0.05t = \ln 2$

$t = 13.8629 = 13.9$ years (to 3 sig fig)

55. $5x = 10^{1.5} \Rightarrow x = \dfrac{10^{1.5}}{5} = 6.32$ (to 3 s.f.)

56. $x = \log 4/\log 10 = 0.60206 = 0.602$ (to 3 s.f.)

57. $(x+3)\log 5 = \log 123$

$(x+3) = \dfrac{\log 123}{\log 5} = 2.989978$

$x = -0.0100$ (to 3 s.f.)

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58. $f(1) = 1 + 1 - 2 + 12 = 12$

$f(2) = 8 + 4 - 4 + 12 = 20$

$f(-3) = -27 + 9 + 6 + 12 = 0$

$(x+3)(x^2 - 2x + 4)$

59. $\log x + \log 8 = 2$ $(2^3 = 8)$
$\log 8x = 2$
$8x = 10^2 = 100$
$x = 100/8 = 12.5$

60. $2^{3x} = 34$
$3x \log 2 = \log 34$
$x = \frac{\log 34}{3\log 2} = 1.6958 = 1.70$ (to 3 s.f.)

61.

logr	0	0.30	0.48	0.60	0.70
logt	-0.22	-0.13	-0.08	-0.04	-0.01

$t = k r^n$
$\log t = \log k r^n$
$\log t = \log k + \log r^n$
$\log t = \log k + n \log r$
$\log t = n(\log r) + \log k$ Plotting $\log t$ against $\log r$ gives a straight line and hence the values satisfy the relationship.
Gradient n and vertical intercept at $\log k$
$\log k = -0.22$ $\therefore$ $k = 0.60 = 0.6$ to 1 d.p.
Gradient $= n =$ 0.21 / 0.7 = 0.3 to 1 d.p.

62. $2^{2x+3} = 2^{2x} \times 2^3 = 8 \times 2^{2x} = 8 \times (2^x)^2$
$8(2^x)^2 - 17 \times (2^x) + 2 = 0$
Let $y = 2^x$
$8y^2 - 17y + 2 = 0$
$(8y - 1)(y - 2) = 0$
$y = ⅛$ or 2
$\therefore$ $2^x = ⅛$ or $2^x = 2$
$2^x = 2^{-3}$ or $x = 1$
$\therefore x = -3$ or 1

63. $f(5) = 125 - 150 + 15 + 10 = 0$ $\therefore$ $(x - 5)$ is a factor
$f(2) = 8 - 24 + 6 + 10 = 0$ $\therefore$ $(x - 2)$ is a factor
$f(-1) = -1 - 6 - 3 + 10 = 0$ $\therefore$ $(x + 1)$ is a factor
$x^3 - 6x^2 + 3x + 10 = (x - 5)(x - 2)(x + 1) = 0$
$x = 2$ or 5 or -1

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64. $= 3^4 + 4.3^3x + 6.3^2x^2 + 4.3x^3 + x^4$
$= 81 + 108x + 54x^2 + 12x^3 + x^4$

65. $= 1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$

66. $= 2^5 + 5.2^4(-x) + 10.2^3(-x)^2 + 10.2^2(-x)^3 + 5.2(-x)^4 + (-x)^5$
$= 32 - 80x + 80x^2 - 40x^3 + 10x^4 - x^5$

67. $a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$

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68. $= 64a^6 + 6.32a^5.3b + 15.16a^4.9b^2 + \ldots$
$= 64a^6 + 576a^5b + 2160a^4b^2 + \ldots$

69. $= 1p^5 + 5.p^4(-2q) + 10p^34q^2 + 10p^2(-8q^3) + \ldots$
$= p^5 - 10p^4q + 40p^3q^2 - 80p^2q^3 + \ldots$

70. $(2y - 3)^3$
$= 1(2y)^3 + 3(2y)^2(-3) + 3(2y)(-3)^2 + 1(-3)^3$
$= 8y^3 - 36y^2 + 54y - 27$

71. $(1 + x)^6 = 1 + 6x + \frac{6.5}{2.1}x^2 + \frac{6.5.4}{3.2.1}x^3 + \ldots$
$= 1 + 6x + 15x^2 + 20x^3 + \ldots$

72. $(1 - 2x)^8 = 1 - 16x + 112x^2 - 448x^3$

73. $(1 + 3x)^7 = 1 + 7(3x) + \frac{7.6}{2.1}(3x)^2 + \ldots$
$= 1 + 21x + 189x^2 + \ldots$

74. $(2 + x^2)^5 =$
$1(2)^5 + 5(2)^4(x^2) + 10(2)^3(x^2)^2 + 10(2)^2(x^2)^3 + 5(2)(x^2)^4 + 1(x^2)^5$
$= 32 + 80x^2 + 80x^4 + 40x^6 + 10x^8 + x^{10}$

75. $(1 + x)(2 - 3x)^3$
$= (1 + x)(1(2)^3 + 3(2)^2(-3x) + 3(2)(-3x)^2 + 1(-3x)^3)$
$= (1 + x)(8 - 36x + 54x^2 - 27x^3)$
$= 8 - 36x + 54x^2 - 27x^3 + 8x - 36x^2 + 54x^3 - 27x^4$
$= 8 - 28x + 18x^2 + 27x^3 - 27x^4$

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76. $(1 - x/2)^6$
$= 1 + 6(-x/2) + \frac{6.5}{2.1}(-x/2)^2 + \frac{6.5.4}{3.2.1}(-x/2)^3 + \ldots$
$= 1 - 3x + \frac{15}{4}x^2 - \frac{5}{2}x^3 + \frac{15}{16}x^4 - \frac{3}{16}x^5 + \frac{1}{64}x^6$

77. $(1 - 5x)^7 = 1 + 7(-5x) + \frac{7.6}{2.1}(-5x)^2 + \ldots$
$= 1 - 35x + 525x^2 + \ldots$

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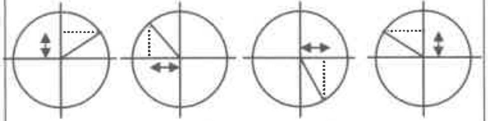

sin30° cos120° cos(– 60°) sin150°

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78. (a) $\cos 135° = -\cos 45°$
(b) $\sin 135° = \sin 45°$ (c) $\cos 280° = \cos 80°$
(d) $\sin 220° = -\sin 40°$ (e) $\sin 315° = -\sin 45°$
(f) $\cos 560° = -\cos 20°$

79. $\sin\theta = 3/5$ $\cos\theta = 4/5$ $\tan\theta = 3/4$

80. $\cos\theta = 5/13$ $\sin\theta = 12/13$ $\tan\theta = 12/5$

81. $\tan\theta = 7/24$ $\sin\theta = 7/25$ $\cos\theta = 24/25$

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82. $\cos\theta = -3/5$ and θ obtuse
$\sin\theta = 4/5$ $\tan\theta = -4/3$

83. $\cos\theta = \frac{15}{17}$ $\tan\theta = \frac{8}{15}$

84. $\sin\theta = -\frac{21}{29}$ $\tan\theta = \frac{21}{20}$

85. $\sin\theta = -\frac{3}{5}$ $\cos\theta = \frac{4}{5}$

86. $\sin\theta = \frac{15}{17}$ $\cos\theta = \frac{8}{17}$ $\sin^2\theta + \cos^2\theta = 1$

87. $\sin 45° = \frac{1}{\sqrt{2}}$ $\cos 45° = \frac{1}{\sqrt{2}}$ $\tan 45° = 1$
$\sin 60° = \frac{\sqrt{3}}{2}$ $\cos 60° = \frac{1}{2}$ $\tan 60° = \sqrt{3}$
$\sin 30° = \frac{1}{2}$ $\cos 30° = \frac{\sqrt{3}}{2}$ $\tan 30° = \frac{1}{\sqrt{3}}$

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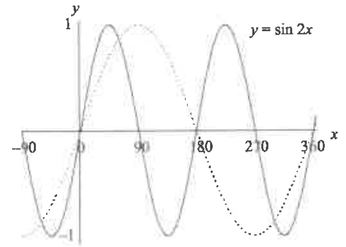
88. (a) $\cos 135° = -\frac{1}{\sqrt{2}}$ (b) $\sin 135° = \frac{1}{\sqrt{2}}$
(c) $\cos 120° = -\frac{1}{2}$ (d) $\sin 210° = -\frac{1}{2}$
(e) $\tan 210° = \frac{1}{\sqrt{3}}$ (f) $\cos 330° = \frac{\sqrt{3}}{2}$

89. (a) $\sin 0° = 0$ (b) $\cos 0° = 1$ (c) $\tan 0° = 0$
(d) $\sin 90° = 1$ (e) $\cos 90° = 0$ (f) $\sin 180° = 0$
(g) $\cos 180° = -1$ (h) $\tan 180° = 0$
(i) $\sin 270° = -1$ (j) $\sin 360° = 0$
(k) $\cos 270° = 0$ (l) $\cos 360° = 1$ (m) $\tan 360° = 0$

90.



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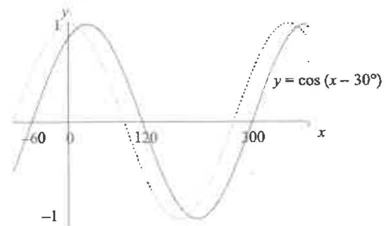

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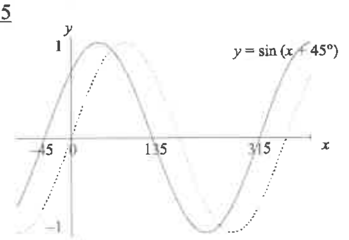
92.



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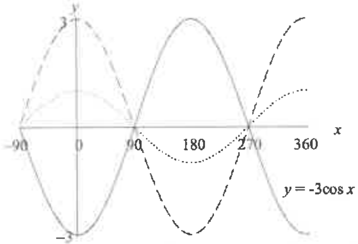

94. $x = \arcsin 0.75 = 48.6°, 131.4°, 408.6°, 491.4°$

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95. $\cos x = -0.6$
$x = -126.9°, 126.9°, 233.1°$ to 1 d.p.

96. $\sin x = 2/5$
$x = 23.6°, 156.4°, 383.6°, 516.4°$ to 1 d.p.

97. $x = 0°, 180°, 360°, 540°, 720°$

98. $\sin A = 0.3$
$A = \arcsin 0.3 = 17.5°$
$A = 2x = 17.5, 162.5, 377.5, 522.5$
$x = 8.7°, 81.3°, 188.7°$(or 8), $261.3°$ to 1 d.p.

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99. $\cos A = 0.2$
$A = \arccos 0.2 = 78.5$
$A = x - 60 = 78.5, 281.5$
$x = 138.5°, 341.5°$ to 1 d.p.

100. $\tan A = 3$
$A = \arctan 3 = 71.6$
$A = 3x = 71.6, 251.6, 431.6$
$x = 23.9°, 83.9°, 143.9°$ to 1 d.p.

101. $\sin A = 0.5$
$A = 30°, 150°, 390°, 510°, 750°$
$x = 60°, 180°, 240°$

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102. Complete: $3\sin x = -5\cos x$
$\frac{\sin x}{\cos x} = -\frac{5}{3}$
$\tan x = -1.66667$
$x = -239.0°, -59.0°, 121.0°, 301.0°$ to 1 d.p.

103. (a) $(2y - 1)(y + 1) = 0$ $\therefore$ $y = -1$ or ½
(b) $1 + \sin x - 2(1 - \sin^2 x) = 0$
$1 + \sin x - 2 + 2\sin^2 x = 0$
$2\sin^2 x + \sin x - 1 = 0$
$\sin x = -1$ or $\sin x = ½$
$x = -90°, 30°, 150°$

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104. $\frac{PR}{\sin 34} = \frac{30}{\sin 98}$
$PR = \frac{30\sin 34}{\sin 98} = 16.9$ to 3 sig.fig
$\frac{QR}{\sin 48} = \frac{30}{\sin 98}$
$QR = \frac{30\sin 48}{\sin 98} = 22.5$ to 3 sig.fig

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105. $\frac{\sin H}{3} = \frac{\sin 40}{4}$
$\sin H = \frac{3\sin 40}{4} = 0.48209$
$\therefore$ angle $H = 28.8°$ to 1 d.p.
angle $G = 180 - (40 + 28.8) = 111.2°$ to 1 d.p.

106. (a) $LM^2 = 25 + 16 - 2 \times 5 \times 4 \times \cos 34$
$= 7.838497$ $\therefore$ $LM = 2.80$ to 3 sig.fig.
(b) Area $= ½ \times 5 \times 4 \times \sin 34$
$= 5.59$ sq. units to 3 sig.fig.

107. $\cos A = \frac{16+25-36}{2\times4\times5} = 0.125$
$\therefore$ $A = 82.8°$ to 1 d.p.

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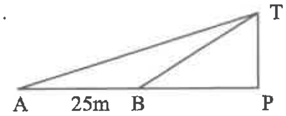
108. $\frac{\sin C}{5.5} = \frac{\sin 20}{2.7}$
$\sin C = \frac{5.5\sin 20}{2.7} = 0.6967$
$C = 44.2°$ or $135.8°$ to 1 d.p.

109. $\frac{PC}{\sin 35} = \frac{600}{\sin 85}$
$PC = \frac{600\sin 35}{\sin 85} = 345$ m to 3 sig.fig.
$\frac{QC}{\sin 60} = \frac{600}{\sin 85}$
$QC = \frac{600\sin 60}{\sin 85} = 522$ m to 3sig.fig

110. $PR^2 = 200^2 + 172^2 - 2 \times 200 \times 172 \times \cos 68$
$= 43811.066$
$PR = 209.31 = 209$ m to 3 sig.fig.
$\cos R = \frac{172^2 + 209.31^2 - 200^2}{2\times172\times209.31} = 0.46380$
$R = 62.4°$ to 1 d.p.
$P = 180 - (68 + 62.4) = 49.6°$ to 1 d.p.

111.



$\frac{BT}{\sin 30} = \frac{25}{\sin 10}$ $\therefore$ $BT = \frac{25\sin 30}{\sin 10} = 71.9846$
$\text{Sin}40 = \frac{height}{BT}$
height $= BT \sin 40 = 71.9846 \times \sin 40$
$= 46.2708 = 46.3$m to 3 sig.fig.

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112. $\sin x = \frac{1}{2}$

$x = -330°, -210°, 30°, 150°$

113. $\cos 2x = 0.2$

$2x = -281.54°, -78.46°, 78.46°, 281.54°$

$x = -140.8°, -39.2°, 39.2°, 140.8°$ to 1 d.p.

114. $\tan x = 1$

$x = 45°, 225°$

115. $\cos^2 x = \frac{1}{2}$

$\cos x = +1/\sqrt{2}$ or $\cos x = -1/\sqrt{2}$

$x = 45, 315$ or $x = 135, 225$

$x = 45°, 135°, 225°, 315°$

116. $2\cos^2\theta + \cos\theta - 1 = 0$

$(2\cos\theta - 1)(\cos\theta + 1) = 0$

$\cos\theta = \frac{1}{2}$ or -1

$\theta = 60°, 300°, 180°$

117. $\cos 30 + \sin 45 = \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} = \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} = \frac{\sqrt{3}+\sqrt{2}}{2}$

118. $(2x + (-1))^5 =$

$1(2x)^5 + 5(2x)^4(-1) + 10(2x)^3(-1)^2 +$

$10(2x)^2(-1)^3 + 5(2x)(-1)^4 + 1(-1)^5$

$= 32x^5 - 80x^4 + 80x^3 - 40x^2 + 10x - 1$

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119. $(1 - 3x)^7$

$= 1 + 7(-3x) + \frac{7.6}{2.1}(-3x)^2 + \frac{7.6.5}{3.2.1}(-3x)^3$

$= 1 - 21x + 189x^2 - 945x^3$

$x = 0.001$

$0.997^7 = 1 - 21(0.001) + 189(.001)^2 - 945(.001)^3$

$= 1 - 0.021 + 0.000189 - 0.000000945$

$= 0.979188$ to 6 d.p.

120. $\sin x = -3\cos x$

$\tan x = -3$

$x = -71.6, 180 - 71.6, 360 - 71.6$

$x = 108.4°, 288.4°$ to 1 d.p.

121. $\frac{AC}{\sin 75} = \frac{25}{\sin 50}$

$AC = \frac{25 \sin 75}{\sin 50}$

Area $= \frac{1}{2} \times 25 \times \frac{25 \sin 75}{\sin 50} \times \sin 55$

$= 323$ m^2 to 3 sig.fig.

122. $PQ = \sqrt{(2-1)^2 + (5-10)^2} = \sqrt{26}$

$QR = \sqrt{(12-2)^2 + (3-5)^2} = \sqrt{104}$

$PR = \sqrt{(12-1)^2 + (3-10)^2} = \sqrt{170}$

$\cos\theta = \frac{170+104-26}{2\times\sqrt{170}\sqrt{104}}$

$\theta = 21.2°$ to 1 dec.pl.

123. $\sin x = -\,^3/_5$

$x = 576.9°, 683.1°$ to 1 dec.pl.

124. $\sin\theta = -\frac{24}{25}$ $\tan\theta = -\frac{24}{7}$

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125. (a) $\frac{dy}{dx} = 3x^2 - 5$ (b) $y = x^3 - x^2 - 20x$

$\frac{dy}{dx} = 3x^2 - 2x - 20$

(c) $y = 2x^{\frac{3}{2}}$ $\frac{dy}{dx} = 3x^{\frac{1}{2}} = 3\sqrt{x}$

(d) $y = 3x^{-2} + \frac{7}{2}x^{-½}$ $\frac{dy}{dx} = -6x^{-3} - \frac{7}{4}x^{-3/2}$

(e) $y = 2x^{3/2} - 5x^{½}$ $\frac{dy}{dx} = 3x^{½} - \frac{5}{2}x^{-½}$

126. $dy/dx = 3x^2 - 12x + 9$

$3(x^2 - 4x + 3) = 3(x-3)(x-1) = 0$

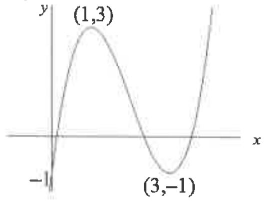
$x = 1$ or 3

$x = 1$ $x = 0.9$ dy/dx is +ve $x = 1.1$ dy/dx is –ve

so $(1, 3)$ is a maximum

$x = 3$ $x = 2.9$ dy/dx is –ve $x = 3.1$ dy/dx is +ve

so $(3, -1)$ is a minimum



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127. $\frac{dy}{dx} = 8x^3 + 10x$ $\frac{d^2y}{dx^2} = 24x^2 + 10$

128. $\frac{dy}{dx} = 10x - 2$ $\frac{d^2y}{dx^2} = 10$

$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 10 + 2(10x - 2) = 20x + 6$

129. $\frac{dy}{dx} = 6x - 3x^2$ For stationary points $\frac{dy}{dx} = 0$

$6x - 3x^2 = 0$

$3x(2 - x) = 0$

$x = 0$ or 2

$\frac{d^2y}{dx^2} = 6 - 6x$

At $x = 0$ $\frac{d^2y}{dx^2} = 6$ i.e + ve

$\therefore$ Min at $(0, 2)$

At $x = 2$ $\frac{d^2y}{dx^2} = -6$ i.e – ve

$\therefore$ Max at $(2, 6)$

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130. $f'(x) = 2x - 5$

$0 = 2x - 5 \Rightarrow x = 2.5$ $f''(x) = 2$

therefore a minimum

for $x < 2.5$ $f'(x) < 0$

for $x = 2.5$ $f'(x) = 0$

for $x > 2.5$ $f'(x) > 0$

therefore $f(x)$ is an increasing function for $x > 2.5$

131. $dy/dx = 3 - 3x^2$

$3(1 - x^2) = 0$ $3(1-x)(1+x) = 0$

$x = 1$ or -1

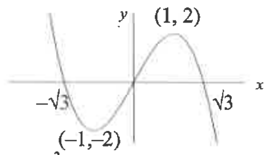
Stationary points $\frac{d^2y}{dx^2} = -6x$

At $x = 1$, $\frac{d^2y}{dx^2} = -6$ negative, so $(1, 2)$ Maximum

At $x = -1$, $\frac{d^2y}{dx^2} = 6$ positive, so $(-1, -2)$ Minimum

$y = x(3 - x^2) = x(\sqrt{3} - x)(\sqrt{3} + x)$

Cuts x axis $(0, 0)$, $(\sqrt{3}, 0)$, $(-\sqrt{3}, 0)$



132.(a) $dy/dx = 3x^2 + 6x - 24 = (3x + 12)(x - 2)$

$= 3(x + 4)(x - 2)$

(b) Grad = 0 for $x = -4$ and $x = 2$

(c) and (d)

At $x = -4$, $y = 85$

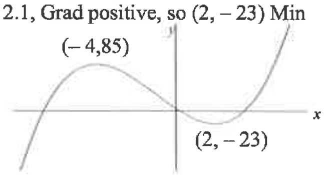
$x = -4.1$, Grad positive

$x = -3.9$, Grad negative, so $(-4, 85)$ Max.

At $x = 2$, $y = -23$

$x = 1.9$, Grad negative

$x = 2.1$, Grad positive, so $(2, -23)$ Min



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133. $dy/dx = 8x^3 - 12x^2 = 4x^2(2x - 3)$

Stationary points at $x = 0$, $y = 5$

and $x = 1.5$, $y = 1.625$

At $x = -0.1$, Grad negative

at $x = 0.1$, Grad negative

So $(0,5)$ point of inflection

At $x = 1.4$, Grad negative

at $x = 1.6$, Grad positive so $(1.5, 1.625)$ is Min.

134. $V = x(10 - 2x)^2 = 100x - 40x^2 + 4x^3$

$dV/dx = 100 - 80x + 12x^2$

$3x^2 - 20x + 25 = 0$

$(3x - 5)(x - 5) = 0$

$x = 5$(not possible) or $^5/_3$

Max volume $= 100\times{}^5/_3 - 40(^5/_3)^2 + 4(^5/_3)^3$

$= 2000/27$ cm^3

135. $V = 144 = 2x^2h$ $\therefore$ $h = 144/2x^2$

$A = 4x^2 + 2xh + 4xh$

$= 4x^2 + 2x(144/2x^2) + 4x(144/2x^2)$

$= 4x^2 + 432x^{-1}$

$dA/dx = 8x - 432x^{-2} = 0$

$8x^3 = 432 \Rightarrow x = 3.77976$

Minimum surface area = 171 to 3 sig.fig.

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136. (a) $\frac{8x^4}{4} + c = 2x^4 + c$

(b) $3\frac{x^2}{2} - 2x + c$

(c) $\int x + \frac{2}{x^2} - \sqrt{x}\,dx = \int x + 2x^{-2} - x^{\frac{1}{2}}\,dx$

$= \frac{x^2}{2} + \frac{2x^{-1}}{-1} - \frac{2x^{\frac{3}{2}}}{3} + c$

$= \frac{x^2}{2} - \frac{2}{x} - \frac{2x^{\frac{3}{2}}}{3} + c$

137. $[7x]_2^5 = [35] - [14] = 21$

138. $[2x^2]_1^3 = [18] - [2] = 16$

139. $\left[\frac{x^3}{3} + 5x\right]_0^2 = \left[\frac{8}{3} + 10\right] - [0] = \frac{38}{3}$

140. $\int 4x^{\frac{1}{2}} - 2x^{-2}dx = \left[4.\frac{2}{3}x^{\frac{3}{2}} - \frac{2x^{-1}}{-1}\right]_1^4 = 17\frac{1}{6}$

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141. Complete: $0 = (3 - x)(3 + x)$ $x = 3$ or -3

$Area = \left[9x - \frac{x^3}{3}\right]_{-3}^3 = [27 - 9] - [-27 + 9] = 36$

142. $\int_0^1 x^2 - x\,dx = \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_0^1 = \frac{1}{3} - \frac{1}{2} = -\frac{1}{6}$

$\int_1^2 x^2 - x\,dx = \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_1^2 = \left[\frac{8}{3} - \frac{4}{2}\right] - \left[\frac{1}{3} - \frac{1}{2}\right] = \frac{5}{6}$

Area $= \frac{1}{6} + \frac{5}{6} = 1$ square unit

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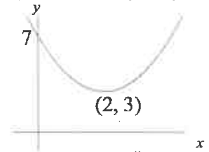
143. $f(x) = (x - 2)^2 - 4 + 7 = (x - 2)^2 + 3$



Area $= \int_0^3 x^2 - 4x + 7dx = \left[\frac{x^3}{3} - 2x^2 + 7x\right]_0^3$

$= [9 - 18 + 21] - [0] = 12$ square units

144. $x^2 + 2 = 11$ $\Rightarrow$ $x = 3$ or -3

Area $= 6 \times 11 - \int_{-3}^3 x^2 + 2\,dx = \left[\frac{x^3}{3} - 2x\right]_{-3}^3$

$= 66 - [(9 + 6) - (-9 - 6)] = 66 - 30$

$= 36$ square units.

145. Area under straight line AB $= \frac{1}{2} \times 2 \times 18 = 18$

Area under curve =

$\int_1^3 x^2 + 5x - 6dx = \left[\frac{x^3}{3} + \frac{5x^2}{2} - 6x\right]_1^3$

$= \left[9 + \frac{5}{2} \times 9 - 18\right] - \left[\frac{1}{3} + \frac{5}{2} - 6\right] = \frac{50}{3}$

Required area $= 18 - 50/3 = {}^4/_3$ square units

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146. (a) $\int x^3 - 6x^2\,dx = [(1/4)x^4 - 2x^3]^1_{-1}$

$= (\frac{1}{4} - 2) - (\frac{1}{4} + 2) = -4$

(b). $\int_{-1}^{1} 1 - t^3\,dt = [t - (1/4)t^4]_{-1}^{1}$

$= (1 - \frac{1}{4}) - (-1 - \frac{1}{4}) = 2$

147. $\int_1^3 \frac{3x}{2} + 2dx = \left[\frac{3x^2}{4} + 2x\right]_1^3 = \frac{27}{4} + 6 - (\frac{3}{4} + 2) = 10$

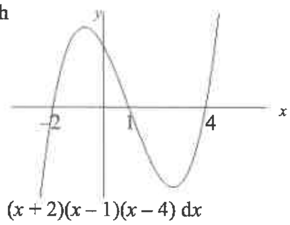
148. Area under curve =

$\int_0^1 x^3 - 5x^2 - x + 5dx = \left[\frac{x^4}{4} - \frac{5x^3}{3} - \frac{x^2}{2} + 5x\right]_0^1$

$= \left[\frac{1}{4} - \frac{5}{3} - \frac{1}{2} + 5\right] - [0] = \frac{37}{12}$

Required area $= \frac{37}{12} - \frac{1}{2} \times 1 \times 5 = \frac{7}{12}$ sq. units

149. sketch



(a) $\int_{-2}^{1} (x + 2)(x - 1)(x - 4)\,dx$

$= \int_{-2}^{1} x^3 - 3x^2 - 6x + 8\,dx$

$= [(1/4)x^4 - x^3 - 3x^2 + 8x]_{-2}^{1}$

$= (25 - 1 - 3 + 8) - (4 + 8 - 12 - 16) = 20.25$ sq units

(b) $\int_1^4 x^3 - 3x^2 - 6x + 8 \, dx$

$= [(1/4)x^4 - x^3 - 3x^2 + 8x]_1^4$

$= (64 - 64 - 48 + 32) - (.25 - 1 - 3 + 8)$

$= -20.25$. So area 20.25 square units (since area has a positive value)

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150 . Magnitude $\sqrt{5^2+2^2} = \sqrt{29}$

$\tan\theta = \frac{2}{5}$ $\quad \theta = 21.8^\circ$

direction is 21.8° with the **i** direction

151. $x = 8\cos 30 = (4\sqrt{3})$ $\quad y = 8\sin 30 = 4$

$(4\sqrt{3})\mathbf{i} + 4\mathbf{j}$ or $6.93\mathbf{i} + 4\mathbf{j}$

152. $x = 20\cos 60 = 10$ $\quad y = 20\sin 60 = (10\sqrt{3})$

$10\mathbf{i} - (10\sqrt{3})\mathbf{j}$ or $10\mathbf{i} - 17.3\mathbf{j}$

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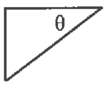
153. Magnitude $= \sqrt{9+4} = \sqrt{13}$



$\tan\theta = \frac{3}{2}$ $\quad \theta = 56.3^\circ$

Direction is 236.3° with the positive **i** direction.

154.

$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$ $\qquad \overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC}$

$\overrightarrow{CB} = \overrightarrow{CO} + \overrightarrow{OB}$ $\qquad \overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC}$

155.

$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$ $\qquad \overrightarrow{BA} = \overrightarrow{BO} + \overrightarrow{OA}$

$= \begin{pmatrix}-2\\-4\end{pmatrix} + \begin{pmatrix}7\\1\end{pmatrix} = \begin{pmatrix}5\\-3\end{pmatrix}$ $\qquad = \begin{pmatrix}-7\\-1\end{pmatrix} + \begin{pmatrix}2\\4\end{pmatrix} = \begin{pmatrix}-5\\3\end{pmatrix}$

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156. $\overrightarrow{DC} = \underline{p}$ $\quad \overrightarrow{BC} = \underline{q}$

$\overrightarrow{BD} = \overrightarrow{BA} + \overrightarrow{AD} = -\underline{p} + \underline{q} = \underline{q} - \underline{p}$

$\overrightarrow{BM} = \frac{1}{2}\overrightarrow{BC} = \frac{1}{2}\underline{q}$ $\quad \overrightarrow{AM} = \overrightarrow{AB} + \overrightarrow{BM} = \underline{p} + \frac{1}{2}\underline{q}$

$\overrightarrow{AN} = \overrightarrow{AD} + \overrightarrow{DN} = \underline{q} + \frac{1}{2}\underline{p}$

$\overrightarrow{MN} = \overrightarrow{MA} + \overrightarrow{AN} = -\left(\underline{p} + \frac{1}{2}\underline{q}\right) + \underline{q} + \frac{1}{2}\underline{p} = \frac{1}{2}\underline{q} - \frac{1}{2}\underline{p}$

157. mag $\sqrt{5^2+12^2} = 13$ $\quad$ unit vector $\frac{5}{13}\mathbf{i} - \frac{12}{13}\mathbf{j}$

158. $\overrightarrow{AB} = 8\mathbf{i} - 3\mathbf{j} - 3\mathbf{i} - 5\mathbf{j} = 5\mathbf{i} - 8\mathbf{j}$

AB $= \sqrt{25+64} = \sqrt{89} = 9.43$ to 3 sig.fig.

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159. PQ $= \sqrt{14^2+2^2} = \sqrt{200}$ $\quad$ QR $= \sqrt{7^2+8^2} = \sqrt{113}$

PR $= \sqrt{7^2+6^2} = \sqrt{85}$

$\cos(\angle PRQ) = \frac{113+85-200}{2\times\sqrt{113}\times\sqrt{85}} = -0.01020$

$\angle PRQ = 90.6^\circ$

160. Even number 2n $\quad$ Odd number 2m – 1
Where n and m are positive integers.
LHS = 2n × (2m – 1) = 2n(2m – 1) which is a multiple of 2. ∴ Even.

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161. Odd number 2n – 1 $\quad$ Even number 2m
Where n and m are positive integers.
LHS = 2m × (2n – 1) = 2m(2n – 1) which is a multiple of 2. ∴ Even.

162. Odd number 2n – 1 and a different odd number 2m – 1
Where n and m are positive integers.
LHS = (2n – 1) × (2m – 1)
= 4mn – 2m – 2n + 1 = 2(2mn – m – n) +1
2(2mn – m – n) is a multiple of 2 and therefore even. Add 1 to an even number and you obtain an odd number. ∴ odd × odd = odd

163. p is odd $p^2 = p \times p$ $\quad$ is odd × odd = odd
∴ $p^2 + p$ = odd + odd = even

164. $(2n-1)^3 - (2n-1)^2 = (2n-1)^2[(2n-1) - 1]$
$= (2n-1)^2(2n-2)$
$= 2(2n-1)^2(n-1)$
Which is a multiple of 2 ∴ even

165. The nth term is 5n – 3
Use 5n – 3 and 5m – 3for two different terms.
Where n and m are positive integers.
Difference = 5n – 3 – (5m – 3) = 5n – 3 – 5m + 3
= 5n – 5m
= 5(n – m) Which is a multiple of 5.

166. $(2n-1)^2 - (2n-1) = 4n^2 - 4n + 1 - 2n + 1$
$= 4n^2 - 6n + 2$
$= 2(2n^2 - 3n + 1)$
i.e. a multiple of 2 ∴ even

167. $n^2 + 4n + 5 = (n+2)^2 - 4 + 5$
$= (n+2)^2 + 1$
The squared term is ≥ 0
So $n^2 + 4n + 5$ is positive for all n.

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168. $2xy \le x^2 + y^2$
$0 \le x^2 + y^2 - 2xy = x^2 - 2xy + y^2$
RHS $= (x-y)^2$
The squared term is always ≥ 0 hence proved.

169. $x = 2$ $\quad x^2 = 4$ $\quad 2^x = 2^2 = 4$
$x = 3$ $\quad x^2 = 9$ $\quad 2^x = 2^3 = 8$
$x = 4$ $\quad x^2 = 16$ $\quad 2^x = 2^4 = 16$ $\quad$ proved.

170. $n = 1$, $\quad 3^n + 2 = 3 + 2 = 5$ true
$n = 5$, $3^n + 2 = 243 + 2 = 245 = 5\times 49$
∴ not prime and so statement not true.

171. n = 2 $\quad$ 1+ 2 = 3 $\quad$ which is not divisible by 2
∴ Statement not true.

172. n = 7 $\quad$ 196 + 7 = 203 = 7 × 29 so not prime.
[notice that n = 7 gives $4\times 7^2 + 7 = 7(4\times 7 + 1)$]

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173. $\overrightarrow{BA} = -(-3\mathbf{i} - 3\mathbf{j}) + 2\mathbf{i} + \mathbf{j} = 5\mathbf{i} + 4\mathbf{j}$

$\overrightarrow{BC} = -(-3\mathbf{i} - 3\mathbf{j}) + (-2\mathbf{i} + 4\mathbf{j}) = \mathbf{i} + 7\mathbf{j}$

$\overrightarrow{AC} = -(2\mathbf{i} + \mathbf{j}) + (-2\mathbf{i} + 4\mathbf{j}) = -4\mathbf{i} + 3\mathbf{j}$

$\cos(\angle ABC) = \frac{41+50-25}{2\times\sqrt{41}\times\sqrt{50}} = 0.728848$

$\angle ABC = 43.2^\circ$ to 1 d.p.

174. $y = x^2 + 54x^{-1}$ $\quad dy/dx = 2x - 54x^{-2} = 0$
$2x = 54/x^2$
$x^3 = 27$
$x = 3$, $y = 9 + 18 = 27$

$\frac{d^2y}{dx^2} = 2 + 108x^{-3}$ which is +ve when $x = 3$
i.e. a minimum at (3, 27)

175. $\overrightarrow{AB} = -\mathbf{i} - 4\mathbf{j} + 5\mathbf{i} - 2\mathbf{j} = 4\mathbf{i} - 6\mathbf{j}$
AB $= \sqrt{16+36} = \sqrt{52} = 2\sqrt{13}$

176. $n + (n+1) + (n+2) + (n+3) = 4n + 6$
4n is divisible by 4 Adding 6 makes it not divisible by 4.

177. $\underline{a} - \underline{b} = 2\mathbf{i} - 3\mathbf{j} - \mathbf{i} + 5\mathbf{j} = \mathbf{i} + 2\mathbf{j}$
$|\underline{a} - \underline{b}| = \sqrt{1+4} = \sqrt{5}$

178. OA $= \sqrt{1+9} = \sqrt{10}$
OB $= \sqrt{4+25} = \sqrt{29}$
$\overrightarrow{AB} = -(\mathbf{i} - 3\mathbf{j}) - 2\mathbf{i} + 5\mathbf{j} = -3\mathbf{i} + 8\mathbf{j}$
AB $= \sqrt{9+64} = \sqrt{73}$
$\cos(\angle AOB) = \frac{10+29-73}{2\times\sqrt{10}\times\sqrt{29}} = -0.99827$
$\angle AOB = 176.6^\circ$

179. $\int_1^{\sqrt{2}} \frac{3}{2}x + 2x^{-3}\,dx = \left[\frac{3}{2\times 2}x^2 + \frac{2x^{-2}}{-2}\right]_1^{\sqrt{2}}$
$= \frac{5}{4}$

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180. x^3 term ${}^6C_3(2x)^3(3)^3$, ${}^6C_3 = (6.5.4)/(3.2.1) = 20$
∴Coefficient 20 × 8 × 27 = 4320

181. $(x+4)^2 + (y-3)^2 = 4$ centre (– 4, 3) radius 2

182. θ = 30°, 330°

183. $\int_0^4 \frac{1}{2}x^{\frac{3}{2}} - 2x^{-\frac{1}{2}}dx = \left[\frac{1}{5}x^{\frac{5}{2}} - 4x^{\frac{1}{2}}\right]_0^4$
$= \left[\frac{32}{5} - 8\right] - [0] = -\frac{8}{5}$

184. $1 - 2(1 - \cos^2\theta) - \cos\theta = 0$
$2\cos^2\theta - \cos\theta - 1 = 0$
$(2\cos\theta + 1)(\cos\theta - 1) = 0$
$\cos\theta = -½$ or 1
θ = 0°, 120°, 240°, 360°

185. $x^2 = 2x$
$x^2 - 2x = 0$ $\quad x(x-2) = 0$ $\quad x = 0$ or 2

Area $= \int_0^2 2x - x^2\,dx = \left[x^2 - \frac{x^3}{3}\right]_0^2$

= 4 – 8/3 – (0) = 4/3 square units

186. (a) f(2) = 16 – 36 + 28 – 8 = 0
∴ $(x-2)$ is a factor of f(x)
(b) f(x) = $(x-2)(2x^2 - 5x + 4)$
For the quadratics ‘$b^2 - 4ac$’ = 25 – 4×2×4 = –7
Which is negative - so no further real roots.

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187. $(1-2x)^7 = (1 + (-2x))^7$
$= 1 + 7(-2x) + \frac{7.6}{2.1}(-2x)^2 + \frac{7.6.5}{3.2.1}(-2x)^3$
$= 1 - 14x + 84x^2 - 280x^3$

188. Angle ABP = 65° $\quad \frac{AP}{\sin 65} = \frac{8}{\sin 70}$

AP $= \frac{8\sin 65}{\sin 70} = 7.71578 = 7.72$ nautical miles

189. $3x^2 - x^3 = 0$
$x^2(3-x) = 0$ $\quad ∴ x = 0$ or 3

Area $= \int_0^3 3x^2 - x^3 dx = \left[3\frac{x^3}{3} - \frac{x^4}{4}\right]_0^3$

= 27 – 81/4 – (0) = 27/4 square units

190. f(3) = 125–150+15+10 = 0
∴ $(x-3)$ is a factor
$x^3 - 19x + 30 = (x-3)(x^2 + 3x - 10) = 0$
$(x-3)(x-2)(x+5) = 0$
$x = 2$ or 3 or –5

191. $P = 1000e^{kt}$
$2000 = 1000e^{2k}$
$2k = \ln 2$ $\quad \Rightarrow \quad k = \ln 2/2$
$P = 1000e^{(\ln 2/2)t}$
$5000 = 1000e^{(\ln 2/2)t}$
$(\ln 2/2)\,t = \ln 5 \Rightarrow t = 2\ln 5/\ln 2$
= 4.64 to 3 sig. fig.

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192. $\cos 3x = 0$
$3x = 90, 270, 450$
∴ $x = 30^\circ, 90^\circ, 150^\circ$

193. Centre $\left(\frac{2+10}{2}, \frac{3+9}{2}\right) = (6, 6)$

radius $= ½ \times \sqrt{(10-2)^2 + (9-3)^2} = ½ \times 10 = 5$
$(x-6)^2 + (y-6)^2 = 25$

194. $[x^4/4 - 5x^3/3 + x] = 4 - 40/3 + 2 = -{}^{22}/_{3}$

195. r + h = 24 ⇒ h = 24 – r
$V = \pi r^2 h = \pi r^2(24 - r) = 24\pi r^2 - \pi r^3$
$dV/dr = 48\pi r - 3\pi r^2 = 0$ $\quad \Rightarrow r = 16$
$d^2V/dr^2 = 48\pi - 6\pi r$
when r = 16, $d^2V/dr^2 = -48\pi$ i.e. –ve ∴ Maximum
Maximum volume = 6144π – 4096π = 2048π

196. (a) $x = \log 43/\log 5 = 2.34$ to 3 sig. fig.
(b) $x - 1 = \ln 50$ $\quad \Rightarrow x = 4.91$ to 3 sig. fig.
(c) $(4x+1)^2 = e^9$ $\Rightarrow x = 22.3$ to 3 sig. fig.

197. $\overrightarrow{AB} = -(9\mathbf{i} - 3\mathbf{j}) - \mathbf{i} + 2\mathbf{j} = -10\mathbf{i} + 5\mathbf{j}$

$\overrightarrow{AP} = {}^2/_5(-10\mathbf{i} + 5\mathbf{j}) = -4\mathbf{i} + 2\mathbf{j}$

$\overrightarrow{OP} = 9\mathbf{i} - 3\mathbf{j} - 4\mathbf{i} + 2\mathbf{j} = 5\mathbf{i} - \mathbf{j}$

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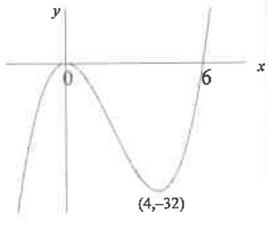
198. (a) $\frac{dy}{dx} = 3x^2 - 12x = 3x(x-4)$
$0 = 3x(x-4)$ $\quad ∴ x = 0$ or 4
Stationary points (0, 0) and (4, –32)
$\frac{d^2y}{dx^2} = 6x - 12$
$x = 0$ $\quad \frac{d^2y}{dx^2} = -12$ ∴ (0, 0) is a maximum
$x = 4$ $\quad \frac{d^2y}{dx^2} = 12$ $\quad$ ∴ (4,–32) is a minimum

(b) $y = x^2(x-6)$ (c)



(d) $0 < x < 4$

199.

t	1	2	3	4
$\log s$	0.243	–0.357	–0.959	–1.523

$s = ka^{-t}$
$\log s = \log ka^{-t}$
$\log s = \log k + \log a^{-t}$
$\log s = \log k - t\log a$
$\log s = -(\log a)t + \log k$ $\quad$ Plotting $\log s$ against t gives a straight line with gradient – log$a$ and vertical intercept at log$k$
Points lie on a straight line and therefore the experimental values satisfy the relationship.
$a = 3.9$ Accept 3.7 to 4.1 , $\quad k = 6.8$ Accept 6 to 8